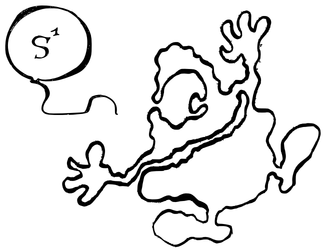
Exotic Aspherical Manifolds

A quick tour through Davis's zoo of aspherical manifolds

Clara Löh

Spring 2009



In a series of three talks, we give an introduction to Davis's construction of peculiar aspherical manifolds and give a quick tour through the resulting zoo of aspherical manifolds.
These notes contain a primer of the subject (describing the basic tools and results), a schedule for the three talks, as well as some illustrative material.

1 A quick tour through the quick tour

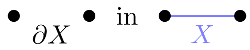
The class of aspherical manifolds plays an important rôle in various fields of topology. For instance, according to the Borel conjecture, aspherical manifolds should be very rigid topological objects. Two natural perspectives on aspherical manifolds are established by the questions "Which manifolds have a contractible universal covering?" and "Which groups admit a model for the classifying space that happens to be a manifold?". The condition for a topological space to be both a manifold and aspherical is very restrictive and for a long time basically the only known examples were the ones given by non-positively curved Riemannian manifolds and fibre bundle constructions.

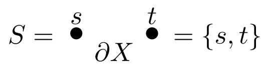
The perception of aspherical manifolds changed drastically when Davis in his landmark paper [2] provided the first construction of exotic aspherical manifolds – more specifically, of closed, aspherical manifolds whose universal covering is not homeomorphic to Euclidean space(!). Subsequently, Davis and others refined this technique to construct closed, aspherical manifolds with exotic fundamental groups and exotic geometric properties [3, 4].

Mirror, mirror on the wall ...

The key idea underlying Davis's construction is to glue copies of an aspherical space along certain subspaces, called mirrors, according to the combinatorics of an abstract reflection group – when the input space and the reflection group are chosen appropriately, the glued space is an aspherical manifold; taking quotients then leads to closed, aspherical manifolds.

In the following, we describe these steps in more detail:

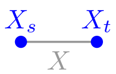


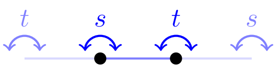


$W = \langle s, t \mid s^2, t^2 \rangle \cong D_\infty$

- *Input.* We start with a compact, aspherical space X together with a closed subspace ∂X, which is triangulated as a flag complex.
- *The associated Coxeter group and mirror structure.* From the flag complex ∂X we can read off the presentation of a group W; namely, we take the set S of vertices as generators (of order 2) and for every simplex of ∂X, we introduce the relation that the product of the vertices of this simplex has order 2. It turns out that (W, S) is an abstract reflection group – more precisely, a right-angled Coxeter group.

  As mirrors on X we choose the family $(X_s)_{s\in S}$ of subspaces X_s of ∂X that are – in a certain sense – fixed under $s \in S \subset W$. More explicitly, for $s \in S$, the space X_s is just the closed star of s in the barycentric subdivision of ∂X.

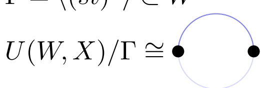




- *The gluing construction.* The fundamental step of the whole construction is gluing copies of X along the mirrors $(X_s)_{s\in S}$ of X via the combinatorics given by the Coxeter group (W, S). The resulting space is denoted by $U(W, X)$, and carries a proper, cocompact W-action.

  At this point it is crucial that we are dealing with a Coxeter group and that the mirror structure on X is also linked to this Coxeter group. In fact, this guarantees that we can write $U(W, X)$ as an increasing union of aspherical pieces and hence that $U(W, X)$ is aspherical.

$\Gamma = \langle (st)^2 \rangle \subset W$



- *Finding a compact quotient.* In general, the space $U(W, X)$ will be non-compact. Because W acts cocompactly on $U(W, X)$, for every subgroup Γ of W of finite index, the quotient $U(W, X)/\Gamma$ is compact. Furthermore, the obvious retraction $U(W, X) \longrightarrow X$ induces a retraction $U(W, X)/\Gamma \longrightarrow X$.

  In particular, if $(X, \partial X)$ is a compact aspherical manifold with boundary and if Γ is a torsion-free subgroup of W of finite index, then $U(W, X)/\Gamma$ is a closed, aspherical manifold that retracts onto X.

Tailoring aspherical manifolds

By adapting certain parameters of the construction described above, we can tailor aspherical manifolds with peculiar properties. Most of these adapted constructions provide examples of the following two types:

– *Aspherical manifolds with exotic fundamental groups.* By thickening finite simplicial models, for every group G of type F we can find a model $(X, \partial X)$ of BG that is a manifold with (triangulated) boundary. Applying the assembly line above to the input $(X, \partial X)$ results in a closed, aspherical manifold that retracts onto X.

  In particular, strange groups of type F give rise to closed, aspherical manifolds with exotic fundamental groups. For example, one can conclude that there exist closed, aspherical manifolds whose fundamental group is not residually finite or whose fundamental group has unsolvable word problem.

– *Aspherical manifolds with exotic geometric properties.* Choosing as inputs aspherical spaces $(X, \partial X)$ with exotic geometric properties, also the closed, aspherical manifolds M resulting from the assembly line above will have exotic geometric properties.

  For instance, starting with a triangulable homology sphere ∂X, we can always find a contractible manifold X whose boundary coincides with ∂X. Then M is a closed, aspherical manifold. If ∂X is not simply connected, then the universal covering (which is $U(W, X)$ in this case) of M is not simply connected at infinity. In particular, if $\dim M > 2$, then M is not covered by Euclidean space.

  Another interesting choice is to take a compact, aspherical manifold $(X, \partial X)$ with triangulated boundary whose Spivak normal fibration does not reduce to a linear vector bundle. Then the Spivak normal fibration of M does not reduce to a linear vector bundle and hence M is an example of a closed, aspherical manifold that is not smoothable.

A comprehensive account of the applications emerging from such constructions is part of Davis's book [4], covering also various consequences for group cohomology as well as the aspect of curvature.

2 Schedule

First session – Reflection groups

Introduction and Overview (*Clara Löh; 15 min.*). A brief introduction into the subject (cf. Section 1) describing the fundamental questions, tools and results, including an overview of the topics of the talks.

Abstract reflection groups (*Christian Siegemeyer; 75 min.*). This talk provides an introduction to abstract reflection groups (i.e., to Coxeter groups) – starting from the basic terminology for Coxeter groups (Coxeter systems, reflection systems, spherical subsets/cosets, ...), then proceeding to basic algebraic properties of Coxeter groups (in particular, cosets of special subgroups), and ending with the complexes associated to Coxeter groups (the nerve, the Davis complex, the Cayley 2-complex).

Probably, these concepts are best accompanied with a few elementary running examples (for instance, the infinite dihedral group, symmetric groups, reflection groups in the model spaces of Riemannian geometry, ...).

Literature. [4, Chapter 2.2, Chapter 3 (Theorem 3.3.4), Chapter 4.1 (Theorem 4.1.6), Chapter 7.1, p. 126] · [1, Chapters I and II]

Second session – Mirrored spaces

Mirrored spaces and the gluing construction (*Christian Wegner; 40 min.*). Definition of mirrored spaces, the gluing construction, the Davis complex from the perspective of mirrored spaces (including the cell structure and examples).

Literature. [4, Chapter 5.1, 5.2, Chapter 7.2–7.4]

Algebraic topology of the gluing construction (*Wolfgang Steimle; 50 min.*). This talk should explain (at least the ideas behind) the computations of the homology, the fundamental group and the fundamental group at infinity of the gluing construction. In particular, the asphericity criterion for the gluing construction should be presented.

Literature. [4, Chapter 8.1, 8.2 (Corollaries 8.2.8, Theorem 8.2.13), Chapter 9 (Theorems 9.1.3, 9.1.5, 9.2.2)]

Third session – Exotic aspherical manifolds

The reflection group trick (*Thilo Küssner; 60 min.*). Description of the reflection group trick and its applications. A selection of applications should be sketched – e.g., the existence of oriented, closed, connected aspherical manifolds with fundamental groups that are not residually finite, that have unsolvable word problem, etc. A short sketch of the proof that finitely generated Coxeter groups are virtually torsion free should be given. Additionally, the consequences of the reflection group trick from the point of view of the isomorphism conjectures could be discussed.

Literature. [4, (Chapter 10.1), Chapter 11.1, 11.2, 11.3?, 11.4?, Chapter 6.12] · [5]

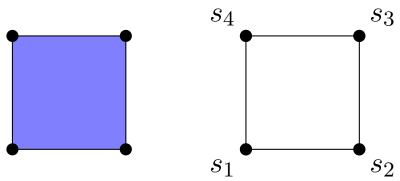
Aspherical manifolds not covered by Euclidean space (*Clara Löh; 30 min.*). This talk should sketch Davis's construction of aspherical manifolds not covered by Euclidean space, including the necessary background on homology spheres (which are a crucial ingredient of this construction).

Literature. [4, Chapter 10.3, 10.5]

3 Guinea pig – growing a torus familiaris

In the following, we illustrate the basic assembly line for aspherical manifolds at a simple example; in particular, this will also present us with an opportunity to get acquainted with the notation involved in the constructions.

As guinea pig we take the unit square $(X, \partial X) := \big([0,1]^2, \partial([0,1]^2)\big)$, which is an aspherical manifold with boundary, and equip ∂X with the "obvious" triangulation as a flag complex.



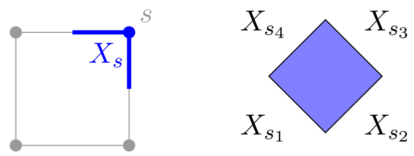
The associated Coxeter group. From the flag complex ∂X we read off the right-angled Coxeter group (W, S) given by $S := \{s_1, \dots, s_4\}$ and

$$W := \big\langle S \bigm| s_1^2, s_2^2, s_3^2, s_4^2,\ (s_1 s_2)^2, (s_2 s_3)^2, (s_3 s_4)^2, (s_4 s_1)^2 \big\rangle.$$

(Notice that indeed the nerve $L(W, S)$ of (W, S) is isomorphic to the flag complex ∂X).

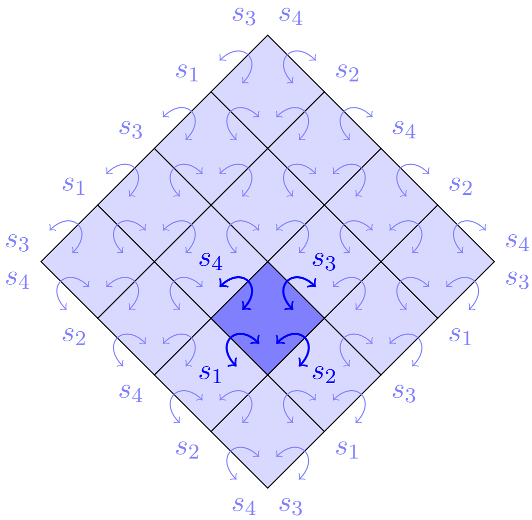
The mirror structure. As next step, we determine the induced mirror structure on X: For $s \in S$ the mirror X_s is the closed star of s in the barycentric subdivision of the boundary $\partial X = L(W, S)$.

For convenience, we now identify $(X, \partial X)$ with the mirrored space obtained from X by applying the (relative) homeomorphism that "pushes the vertices of the square towards the interior."



The gluing construction. The fundamental step of the whole construction is gluing copies of X along the mirrors $(X_s)_{s \in S}$ of X via the combinatorics given by the Coxeter group (W, S); this results in the aspherical(!)

space $U(W, X)$. Notice that $U(W, X)$ is a manifold equipped with an obvious proper, cocompact W-action.



Finding a compact quotient. In order to obtain a compact aspherical manifold, we form the quotient of $U(W, X)$ by a torsion-free subgroup of W of finite index: Because (W, S) is a right-angled Coxeter group, its commutator subgroup Γ is a torsion-free subgroup of finite index. It is not difficult to see that Γ is the (normal) subgroup of W generated by $[s_1, s_3]$ and $[s_2, s_4]$; moreover, a straightforward computation shows that these two elements commute and that $\Gamma \cong \mathbf{Z} \times \mathbf{Z}$. So the quotient $U(W, X)/\Gamma$ is the torus obtained from the sixteen copies of X depicted above by gluing opposite edges of the large square.

The retraction $U(W, X)/\Gamma \longrightarrow X$ is the one induced from the obvious retraction $U(W, X) \longrightarrow X$ given by folding $U(W, X)$ onto X.

Exercise. In this example, how does the Davis complex $\Sigma(W, S)$, i.e., the simplicial complex associated to the poset of all cosets of spherical subsets of (W, S), look like? (A subset of S is *spherical* if it generates a finite subgroup in W.)

Exercise. Carry out this construction for the thickening $[0, 1]^2 \setminus [1/3, 2/3]^2$ of S^1 in $\mathbf{R}^2$ (with the obvious triangulation of the boundary).

4 Cheat sheet – Ubiquitous complexes

$L(W,S)$ [for a Coxeter system (W,S)]
the *nerve* of the Coxeter system (W,S), i.e., the simplicial complex associated with the poset of all non-empty spherical subsets of S [4, p. 123].
If (W,S) is a right-angled Coxeter system, then $L(W,S)$ is a flag complex; conversely, every flag complex is of this form.

$K(W,S)$ [for a Coxeter system (W,S)]
the simplicial complex associated with the poset of all spherical subsets of S [4, p. 126]; hence, $K(W,S)$ is the cone of the barycentric subdivision of $L(W,S)$.

$\Sigma(W,S)$ [for a Coxeter system (W,S)]
the *Davis complex* of the Coxeter system (W,S), i.e., the simplicial complex associated with the poset of all W-cosets of spherical subgroups of W [4, p. 126].
The Davis complex $\Sigma(W,S)$ is contractible and a model for the classifying space of proper W-actions [4, Theorems 8.2.13 and 12.3.4]. In a certain sense, $\Sigma(W,S)$ serves as a replacement for the model spaces $\mathbf{R}^n$, S^n, $\mathbf{H}^n$ for the classical case of geometric reflection groups.
Notice that in the theory of buildings the notation $\Sigma(W,S)$ is used for the simplicial complex associated with the poset of all W-cosets of *all* subgroups of W generated by subsets of S.

$U(G,X)$ [for a mirrored space X with mirrors $(X_s)_{s\in S}$ and a corresponding family $(G_s)_{s\in S}$ of subgroups of G]
the gluing construction of X [4, p. 64].
The most prominent case is that of $U(W,X)$ where X has a closed subspace ∂X that is triangulated as a flag complex and (W,S) is the associated right-angled Coxeter group (hence, ∂X coincides with $L(W,S)$ and the mirrors $(X_s)_{s\in S}$ and the family of subgroups of W are chosen appropriately) [4, p. 66, 212]. Notice that $U(W,K(W,S)) \cong \Sigma(W,S)$ for all Coxeter systems (W,S) (where we take $L(W,S) \subset K(W,S)$ as the triangulated subspace).

References

[1] K. Brown, *Buildings*. Springer, 1989. A textbook on the theory of buildings; the basic components of buildings are Coxeter complexes, and this book provides also a thorough introduction into Coxeter groups.

[2] M. Davis, Groups generated by reflections and aspherical manifolds not covered by Euclidean space. *Ann. of Math. (2)*, 117, pp. 293–324, 1983. Davis's landmark article on the construction of exotic aspherical manifolds paving the way to the various other examples of bizarre aspherical manifolds.

[3] M. Davis, Exotic aspherical manifolds. *Topology of high-dimensional manifolds* (Trieste, 2001), pp. 371–404, volume 9 of *ICTP Lect. Notes*, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002. A concise survey on the construction of exotic aspherical manifolds via reflection groups (using the language of cubical cell complexes).

[4] M. Davis, *The Geometry and Topology of Coxeter Groups*. Volume 32 of *London Mathematical Society Monographs*, Princeton University Press, 2008. Probably the most up-to-date and most comprehensive account of the topic of exotic aspherical manifolds and the related geometry of Coxeter groups.

[5] W. Lück, Survey on aspherical manifolds. Preprint, 2009. Available online at arXiv:0902.2480. A general survey on aspherical manifolds (with a view towards the isomorphism conjectures).

Clara Löh

Adresse: Fachbereich Mathematik und Informatik der WWU Münster
Einsteinstr. 62
48149 Münster

email: clara.loeh@uni-muenster.de
URL: http://wwwmath.uni-muenster.de/u/clara.loeh