Simplicial volume and L^2-Betti numbers

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Abstract. Gromov posed the following question, suggesting a deep connection between two at first glance quite different topological invariants: *If the simplicial volume of an aspherical oriented closed connected manifold is zero, is then also the Euler characteristic of this manifold zero?* Using L^2-Betti numbers, which compute the Euler characteristic, Schmidt showed that when the simplicial volume is replaced by the integral foliated simplicial volume, the corresponding question is answered affirmatively. Hence it is interesting to study the relation between the simplicial volume and the integral foliated simplicial volume of aspherical manifolds. In this talk, I will present some of the recent results on the integral foliated simplicial volume.

1 The name of the game

In the context of geometric group theory, Gromov posed the following question [5; p. 232], suggesting a deep connection between two at first glance completely different invariants:

Question (1.1). *Let M be an aspherical oriented closed connected manifold whose simplicial volume is zero. Does then also the Euler characteristic of M vanish?*

The simplicial volume $\|M\|$ can be thought of as a topological approximation of the Riemannian volume of M (see Section 2).

The Euler characteristic $\chi(M)$ on the other hand is the alternating sum of the number of cells in a given cellular decomposition of M. More precisely,

$$\chi(M) = \sum_{k=0}^{\dim M} (-1)^k \cdot \#k\text{-dimensional cells of } M.$$

One of the early achievements of algebraic topology was to prove that this number indeed only depends on the space at hand, but not on the choice of a cellular decomposition. This fact can, for example, be derived from the equation

$$\chi(M) = \sum_{k=0}^{\dim M} (-1)^k \cdot \dim_{\mathbf{R}} H_k(M, \mathbf{R}).$$

Here, $b_k(M) := \dim_{\mathbf{R}} H_k(M, \mathbf{R})$ is called the k-th Betti number of M. However, these numbers do not vanish very often and hence are not very well suited for approaching the above question.

Gromov suggested to use L^2-Betti numbers instead, an equivariant generalisation of Betti numbers (Section 3), which are zero in a large number of cases. Analogously to ordinary Betti numbers, they can be used to compute the Euler characteristic via

$$\chi(M) = \sum_{k=0}^{\dim M} (-1)^k \cdot b_k^{(2)}(M).$$

Thus, we are led to the following question:

Question (1.2). *Let M be an aspherical oriented closed connected manifold whose simplicial volume is zero. Do then also all L^2-Betti numbers of M vanish?*

(If the Singer Conjecture is true, Questions (1.1) and (1.2) are even equivalent.)

The main motivation for studying the Questions (1.1) and (1.2) is not only to get new vanishing results for the Euler characteristic and L^2-Betti numbers, but also to understand the underlying mechanism relating the simplicial volume and the Euler characteristic.

In order to study Question (1.2), Gromov proposed [6; Section 5.38] to relate L^2-Betti numbers to an invariant (the so-called integral foliated simplicial volume) similar to the simplicial volume and then to show that the simplicial volume and the integral foliated simplicial volume of aspherical manifolds coincide. The first part was worked out by Schmidt [12], the second part is currently investigated.

The rest of these notes is organised as follows. Section 2 contains an introduction to the simplicial volume, Section 3 is an overview over L^2-Betti numbers, and Section 4 discusses Questions (1.1) and (1.2) in more detail. Finally, Section 5 provides a survey on the integral foliated simplicial volume, including recent results shading some light on Questions (1.1) and (1.2).

2 Simplicial Volume

The simplicial volume is a topological invariant, measuring the complexity of (the fundamental class of) a manifold. It was first introduced by Gromov in his proof of the Mostow rigidity theorem. Most of the results on simplicial volume can be found in Gromov's groundbreaking paper *Volume and bounded cohomology* [4].

Singular homology $H_k(\,\cdot\,, R)$ of topological spaces is an algebraic tool encoding information on "generalised k-dimensional triangulations" of certain parts of the spaces.

If M is an oriented closed connected manifold of dimension n, then

$$H_n(M, \mathbf{Z}) \cong \mathbf{Z}.$$

Let $[M]$ be a generator of $H_n(M, \mathbf{Z})$ (a so-called **fundamental class** of M) and let $[M]_{\mathbf{R}} \in H_n(M, \mathbf{R})$ be its image under the change of coefficients homomorphism $H_n(M, \mathbf{Z}) \longrightarrow H_n(M, \mathbf{R})$. The cycles representing $[M]$ or $[M]_{\mathbf{R}}$ (so-called **fundamental cycles**) can be thought of as generalised triangulations of the manifold M.

Definition (2.1). Let M be an oriented closed connected manifold of dimension n. Then

$$\|M\|_{\mathbf{Z}} := \inf\Big\{\sum_{j\in J} |a_j| \;\Big|\; \sum_{j\in J} a_j \cdot \sigma_j \in C_n(M, \mathbf{Z}) \text{ is a fundamental cycle}\Big\} \in \mathbf{Z}$$

is the **integral simplicial volume** of M, and

$$\|M\| := \inf\Big\{\sum_{j\in J} |a_j| \;\Big|\; \sum_{j\in J} a_j \cdot \sigma_j \in C_n(M, \mathbf{R}) \text{ is a fundamental cycle}\Big\} \in \mathbf{R}$$

is the **simplicial volume** of M. ◇

2.1 Properties of the simplicial volume

The integral simplicial volume tends to behave very badly (mainly because there is no division in $\mathbf{Z}$ available), but the simplicial volume has a couple of nice features (see the list below). Unfortunately, to investigate Questions (1.1) and (1.2) a better understanding of the integral simplicial volume seems to be essential.

1. The (integral) simplicial volume is a homotopy invariant.
2. The inclusion $C_*(M, \mathbf{Z}) \subset C_*(M, \mathbf{R})$ of chain complexes shows that $\|M\| \leq \|M\|_{\mathbf{Z}}$. Obviously, in general $\|M\| \neq \|M\|_{\mathbf{Z}}$.
3. The simplicial volume can be computed via bounded cohomology [4, 2].

4. If $f\colon N \longrightarrow M$ is a continuous map of oriented closed connected manifolds of the same dimension, then
$$\|N\| \geq |\deg f| \cdot \|M\|\,.$$
In particular, $\|S^n\| = 0$ *and* $\|(S^1)^n\| = 0$ *for all* $n \in \mathbf{N}_{>0}$.

5. If there is a d-sheeted covering $N \longrightarrow M$ of oriented closed connected manifolds, then
$$\|N\| = d \cdot \|M\|\,.$$

6. (*Proportionality principle* [4, 15, 14]) If M and N are Riemannian oriented closed connected manifolds with *isometric* universal Riemannian coverings, then
$$\frac{\|M\|}{\operatorname{vol} M} = \frac{\|N\|}{\operatorname{vol} N}.$$
In particular, the simplicial volume of flat manifolds is zero.

If M is a hyperbolic n-manifold, then
$$\|M\| = \frac{\operatorname{vol} M}{v_n},$$
where v_n is the maximal volume of geodesic n-simplices in (the closure of) hyperbolic n-space [4, 15, 2, 11]. *In particular, the simplicial volume is not always zero.*

7. For all smooth oriented closed connected n-manifolds [4]
$$\|M\| \leq (n-1)^n \cdot n! \cdot \operatorname{minvol} M,$$
where the minimal volume of M is given by
$$\operatorname{minvol} M := \inf\bigl\{\operatorname{vol}(M,g) \bigm| g \text{ is a Riemannian metric on } M \text{ with } |\sec(g)| \leq 1\bigr\}.$$
In particular, if M admits a non-trivial S^1-action, then $\|M\| = 0$ [16].

8. Since bounded cohomology of amenable groups is trivial [4, 7], if the fundamental group $\pi_1(M)$ is amenable, then $\|M\| = 0$.

9. The simplicial volume is compatible with connected sums and products (up to constants depending only on the dimension) [4, 2]:
$$\|M\| \cdot \|N\| \leq \|M \times N\| \leq c(\dim M + \dim N) \cdot \|M\| \cdot \|N\|\,,$$
and (if $\dim M = \dim N \geq 3$) [4]
$$\|M + N\| = \|M\| + \|N\|\,.$$

Items 6. and 7. show that the simplicial volume is a topological approximation of the (Riemannian) volume.

The proofs of 6., 8., 9. rely on the tight connection of the simplicial volume and bounded cohomology and computations in bounded cohomology.

3 L^2-Betti numbers

Ordinary Betti numbers in some sense count the number of "holes" of topological spaces, indexed by their dimension. However, sometimes it proves useful to transfer topological invariants to an equivariant setting, i.e., to the category of topological spaces with a group action. In this equivariant setting, L^2-Betti numbers can be defined – as an equivariant translation of ordinary Betti numbers.

In the following, we only consider the particularly accessible class of equivariant spaces given by universal coverings of topological spaces equipped with the action by the fundamental group.

Recall that if M is a topological space with fundamental group G, the homology of $\widetilde{M}$ with local coefficients in the right $\mathbf{Z}G$-module R, denoted by $H_*^G(\widetilde{M}, R)$, is the homology of the twisted chain complex

$$C_*^G(\widetilde{M}, R) := R \otimes_{\mathbf{Z}G} C_*(\widetilde{M}, \mathbf{Z}).$$

For example, homology of $\widetilde{M}$ with local coefficients in the trivial G-module $\mathbf{R}$ is just homology of M with real coefficients. On the other hand, homology of $\widetilde{M}$ with local coefficients in $\mathbf{R}G$ coincides with the homology of $\widetilde{M}$ with real coefficients.

Thus a first try for an equivariant Betti number might look like

$$\text{“}\dim_{\mathbf{R}G}\text{”} H_*^G(\widetilde{X}, \mathbf{R}G).$$

However, it turns out that the ring $\mathbf{R}G$ does not admit an appropriate dimension function "$\dim_{\mathbf{R}G}$." Using functional analysis, one can find a ring containing $\mathbf{R}G$ and admitting such a dimension function, namely the group von Neumann algebra $\mathcal{N}G$.

Definition (3.1). Let G be a discrete countable group. Then the algebra of G-equivariant bounded operators from $\ell^2(G, \mathbf{C})$ to $\ell^2(G, \mathbf{C})$ is a von Neumann algebra, the so-called **group von Neumann algebra** $\mathcal{N}G$ of G. ◇

Remark (3.2). Let G be a discrete countable group and let $\varrho \colon \mathbf{C}G \longrightarrow B(\ell^2(G, \mathbf{C}))$ be given by multiplication from the right. Then the (weak) closure of $\varrho(\mathbf{C}G) \subset B(\ell^2(G, \mathbf{C}))$ coincides with $\mathcal{N}G$. ◇

Group von Neumann algebras are tame von Neumann algebras in the sense that they are of type I (if G is finite) or of type II (if G is infinite) and hence give rise to a trace, which in turn can be used to define the dimension of finitely generated projective $\mathcal{N}G$-modules. Astonishingly, this dimension function can be extended to the class of *all* modules over $\mathcal{N}G$ [10].

Definition (3.3). Let M be a topological space with fundamental group G and universal covering $\widetilde{M} \longrightarrow M$, and let $k \in \mathbf{N}$. Then the **k-th L^2-Betti number** of M is defined as

$$b_k^{(2)}(M) := \dim_{\mathcal{N}G} H_k^G(\widetilde{M}, \mathcal{N}G) \in \mathbf{R}. \qquad ◇$$

Like Betti numbers also L^2-Betti numbers have a geometrical interpretation, in terms of the long time behaviour of the heat kernel: Namely, if M is a Riemannian manifold, then

$$\dim_{\mathbf{R}} H_k(M, \mathbf{R}) = \lim_{t \to \infty} \int_M \operatorname{tr}_{\mathbf{C}}\big(e^{-t \cdot \Delta_k}(x,x)\big)\, d\operatorname{vol}_M(x),$$

where $e^{-t\cdot\Delta_k}(x,y)$ is the heat kernel on the L^2-integrable k-forms of M.

Similarly,

$$b_k^{(2)}(M) = \lim_{t \to \infty} \int_F \operatorname{tr}_{\mathbf{C}}\big(e^{-t \cdot \widetilde{\Delta}_k}(x,x)\big)\, d\operatorname{vol}_{\widetilde{M}}(x),$$

where F is a fundamental domain for the action of the fundamental group $\pi_1(M)$ on $\widetilde{M}$ and where $e^{-t\cdot\widetilde{\Delta}_k}(x,y)$ is the heat kernel on the L^2-integrable k-forms of $\widetilde{M}$. I.e., the k-th L^2-Betti number measures the size of the kernel of $\widetilde{\Delta}_k$. Actually, this analytic description was the original definition of L^2-Betti numbers by Atiyah [1].

3.1 Properties of L^2-Betti numbers

Proofs (or at least references to proofs) of the properties listed below can be found in Lück's book on L^2-invariants [10].

1. The L^2-Betti numbers are homotopy invariant.
2. The zeroth L^2-Betti number is determined by the fundamental group via
$$b_0^{(2)}(M) = \frac{1}{|\pi_1(M)|}.$$
3. (*Euler Poincaré formula*) The Euler characteristic can be computed by L^2-Betti numbers, namely
$$\chi(M) = \sum_{k=0}^{\dim M} (-1)^k \cdot b_k^{(2)}(M).$$
In general, this is the only relation between ordinary Betti numbers and L^2-Betti numbers.
4. (*Poincaré duality*) If M is an oriented closed connected n-manifold, then
$$b_k^{(2)}(M) = b_{n-k}^{(2)}(M)$$
for all $k \in \mathbf{N}$.
5. If M is simply connected, then the L^2-Betti numbers coincide with the ordinary Betti numbers.
6. If there is a d-sheeted covering $N \longrightarrow M$ of oriented closed connected manifolds, then
$$b_k^{(2)}(N) = d \cdot b_k^{(2)}(M)$$
for all $k \in \mathbf{N}$.

7. (*Proportionality principle*) If M and N are two Riemannian oriented closed connected manifolds with *isometric* universal Riemannian coverings, then for all $k \in \mathbf{N}$,
$$\frac{b_k^{(2)}(M)}{\operatorname{vol} M} = \frac{b_k^{(2)}(N)}{\operatorname{vol} N}.$$
In particular, the L^2-Betti numbers of flat manifolds are all zero.

8. If G and H are measure equivalent (countably infinite) groups, then there exists a $c \in \mathbf{R}_{>0}$ such that for all $k \in \mathbf{N}$
$$b_k^{(2)}(BG) = c \cdot b_k^{(2)}(BH).$$

9. Let M be an aspherical manifold whose fundamental group contains an infinite amenable normal subgroup. Then all L^2-Betti numbers of M are zero.

10. Let M be an aspherical manifold with non-trivial S^1-action. Then all L^2-Betti numbers of M are zero.

11. If the L^2-Betti numbers of M and N are zero, then also the L^2-Betti numbers of the product $M \times N$ are zero. Moreover, there are explicit formulas expressing the L^2-Betti numbers of products and connected sums in terms of the L^2-Betti numbers of the factors and summands respectively.

4 Comparing simplicial volume and L^2-Betti numbers

4.1 Status of Questions (1.1) and (1.2)

Some reasons to expect Gromov's questions to be answered affirmatively.

- Comparing the results of Subsection 2.1 and 3.1 gives some evidence (but no conceptual reason yet) why Questions (1.1) and (1.2) might have positive answers.

- Moreover, explicit computations show that the L^2-Betti numbers of aspherical oriented closed connected 3-manifolds (satisfying Thurston geometrisation) are all equal to zero [9], i.e., that these manifolds satisfy the Singer conjecture. In particular, Questions (1.1) and (1.2) are answered positively in this case.

Some variations of Gromov's questions.

- Of course, the asphericity condition cannot be easily dropped: For example $\|S^2\| = 0$, but $b_0^{(2)}(S^2) = 1 \neq 0$.

- The converse of the conjecture does not hold. That it is wrong in odd dimensions is predicted by the Singer conjecture. However, it is also possible to construct counterexamples in even dimensions: Let M be a hyperbolic oriented closed connected manifold fibering over the circle S^1 (such manifolds exist for example in dimension 3). Then the product $M \times M$ has non-zero simplicial volume and vanishing L^2-Betti numbers.

- Moreover, it seems essential that the spaces involved are manifolds (or more precisely, possess a Poincaré duality isomorphism): Namely, one can find groups G such that $\chi(G) \neq 0$ (in fact, the absolute value of the Euler characteristic can be made arbitrarily large) and such that $\|\alpha\|_1 = 0$ for all $\alpha \in H_*(G, \mathbf{R})$ [8].

Some consequences if Gromov's questions are answered affirmatively.

A connected subset $Y \subset X$ is amenable, *if the image of the homomorphism $\pi_1(Y) \to \pi_1(X)$ induced by the inclusion is an amenable group. A subset $Y \subset X$ is* amenable, *if all its connected components are amenable.*

- If Gromov's conjecture is true, then the following result of Ivanov [7] could be carried over to L^2-Betti numbers of aspherical manifolds:

  If M is an oriented closed connected n-manifold admitting an amenable covering such that any point of M is contained in at most n sets of this covering, then $\|M\| = 0$.

  Recently, Sauer gave a direct proof of the corresponding result for L^2-Betti numbers [13].

- If M is an oriented closed connected manifold admitting a selfmap of non-trivial degree, then obviously $\|M\| = 0$. If Gromov's conjecture is true, then also the L^2-Betti numbers of aspherical oriented closed connected manifolds with selfmaps of non-trivial degree would be zero. At the time of writing this is only known in special cases [10; Theorem 14.40].

4.2 Comparing the integral simplicial volume and Betti numbers

A first step towards a conceptual reason for a connection between simplicial volume and L^2-Betti numbers (and hence between simplicial volume and Euler characteristic) is the following observation [10], which clearly would be false when replacing the integral simplicial volume by the simplicial volume:

Proposition (4.1). *Let M be an oriented closed connected n-dimensional manifold. Then*

$$\sum_{k=0}^{n} \dim_{\mathbf{R}} H_k(M, \mathbf{R}) \leq (n+1) \cdot \|M\|_{\mathbf{Z}}.$$

The key to the proof of this proposition is Poincaré duality.

Proof. The proof is essentially the same as in Lück's book – we merely count simplices a little more carefully:

Let $\sum_{j=0}^{m} a_j \cdot \sigma_j \in C_n(M, \mathbf{Z})$ be a fundamental cycle of M with $a_j \neq 0$. For $k \in \{0, \ldots, n\}$, we consider the set

$$S_k := \big\{ {}_k\lfloor \sigma_j \bigm| j \in \{1, \ldots, m\} \big\}$$

of all k-back faces of the simplices $\sigma_1, \dots, \sigma_m$. Poincaré duality yields

$$\dim_{\mathbf{R}} H_k(M, \mathbf{R}) \leq |S_k|.$$

Proof. Let $\varphi = [f] \in H^k(M, \mathbf{R})$ with $f|_{S_k} = 0$. By definition of the cap product, we obtain

$$\varphi \cap [M] = \Big[\sum_{j=1}^{m} a_j \cdot f({}_k\lfloor\sigma_j) \cdot \sigma_j\rfloor_{n-k}\Big] = [0].$$

Since $\cdot \cap [M]$ is injective by Poincaré duality, we obtain $\varphi = 0 \in H^k(M, \mathbf{R})$. Therefore, $\dim_{\mathbf{R}} H_k(M, \mathbf{R}) = \dim_{\mathbf{R}} H^k(M, \mathbf{R})$ is bounded by $|S_k|$. □

On the other hand, we clearly have $|S_k| \leq m \leq \sum_{j=0}^m |a_j|$, which gives the result. □

Using equivariant Poincaré duality, the same argument can be applied in the setting of L^2-Betti numbers [12]:

Theorem (4.2). *Let M be an oriented closed connected n-manifold. Then*

$$\sum_{k \in \mathbf{N}} b_k^{(2)}(M) \leq (n+1) \cdot \|M\|_{\mathbf{Z}}.$$

However, since the integral simplicial volume is too rigid, we have to use a larger but not too large coefficient ring instead of $\mathbf{Z}$. In the next section, we introduce a candidate for such a ring, using measure theory and the group action of the fundamental group on the universal covering.

5 The integral foliated simplicial volume

Based on ideas of Gromov [6; Section 5.38], Schmidt [12] defined the integral foliated simplicial volume as follows:

A standard G-space is a standard probability space with a measure preserving G-action by Borel automorphisms

Definition (5.1). Let M be an oriented closed connected n-manifold with fundamental group G. Then the **integral foliated simplicial volume** of M is defined by

$$|M| := \inf\big\{ |M|^X \bigm| [X] \text{ is an isomorphism class of standard } G\text{-spaces}\big\},$$

where for any standard G-space (X, μ)

$$|M|^X := \inf\Big\{ \sum_{j\in J} \int_X |f_j|\, d\mu \Bigm| \sum_{j \in J} f_j \otimes \sigma_j \in L^\infty(X, \mathbf{Z}) \otimes_{\mathbf{Z}G} C_*(\widetilde{M}, \mathbf{Z}) \text{ is a fundamental cycle of } M\Big\}.$$ ◇

The cycles used in the definition of $\|M\|$ and $\|M\|_{\mathbf{Z}}$ respectively can be seen as generalised triangulations of M. Similarly, it is convenient (yet not entirely correct) to think of the fundamental cycles in the twisted chain complex $L^\infty(X,\mathbf{Z}) \otimes_{\mathbf{Z}G} C_*(\widetilde{M},\mathbf{Z})$ as G-equivariant triangulations of $X \times \widetilde{M}$ or as triangulations of a G-fundamental domain of $X \times \widetilde{M}$.

Moreover, the ring $L^\infty(X,\mathbf{Z})$ turns up naturally as a subring of the von Neumann algebra $\mathcal{N}\mathcal{R}_X$ (which is in this case also of type II) corresponding to the orbit equivalence relation $\mathcal{R}_X$ of G acting on X, i.e., the associated group measure space construction $L^\infty(X,\mathbf{Z}) \rtimes G$.

The measure spaces involved provide us with some flexibility and allow us to "divide" some cycles, as we are used to in the world of real or rational coefficients. Hence the integral foliated simplicial volume is not as rigid as the integral simplicial volume. However, it is rigid enough to admit the following estimate:

Theorem (5.2). *Let M be an oriented closed connected manifold of dimension n with fundamental group G. Then*

$$\sum_{k\in\mathbf{N}} b_k^{(2)}(M) \leq (n+1) \cdot \mathbf{|} M \mathbf{|}^X$$

holds for all standard G-spaces. In particular,

$$\sum_{k\in\mathbf{N}} b_k^{(2)}(M) \leq (n+1) \cdot \mathbf{|} M \mathbf{|}\,.$$

Proof. The proof given by Schmidt [12] relies on the fact that the L^2-Betti numbers of M can also be computed using the von Neumann algebra $\mathcal{N}\mathcal{R}_X$ of the standard equivalence relation $\mathcal{R}_X$ induced by the G-action on X (instead of the group von Neumann algebra $\mathcal{N}G$) [12, 3], and then takes advantage of Poincaré duality as in the proof of the estimate $\sum_{k\in\mathbf{N}} \dim_{\mathbf{R}} H_k(M,\mathbf{R}) \leq \|M\|_{\mathbf{Z}}$ for ordinary Betti numbers and the integral simplicial volume (Theorem 4.2). Notice that in this situation the Poincaré duality isomorphism is induced by the cap product

$$\begin{aligned}\hom_{\mathbf{Z}G}\bigl(C_*(\widetilde{M},\mathbf{Z}),\mathcal{N}\mathcal{R}_X\bigr) \otimes_{\mathbf{Z}} L^\infty(X,\mathbf{Z}) \otimes_{\mathbf{Z}G} C_n(\widetilde{M},\mathbf{Z}) &\longrightarrow \mathcal{N}\mathcal{R}_X \otimes_{\mathbf{Z}G} C_{n-*}(\widetilde{M},\mathbf{Z})\\ \varphi \otimes f \otimes \sigma &\longmapsto \overline{\varphi({}_*\lfloor\sigma)\cdot f} \otimes \sigma\rfloor_{n-*},\end{aligned}$$

and that

$$\dim_{\mathcal{N}\mathcal{R}_X} \mathcal{N}\mathcal{R}_X \cdot f \leq \dim_{\mathcal{N}\mathcal{R}_X} \mathcal{N}\mathcal{R}_X \cdot \chi_{\mathrm{supp}(f)} = \mu\bigl(\mathrm{supp}(f)\bigr) \leq \int_X |f|\,d\mu$$

holds for all $f \in L^\infty(X,\mathbf{Z})$. ☐

In view of Question (1.1) and (1.2) it is now natural to investigate under which conditions the integral foliated simplicial volume and the simplicial volume coincide. In particular:

Question (5.3). *Let M be an aspherical oriented closed connected manifold. Does this imply $\|M\| = \mathbf{|} M \mathbf{|}$? Or does at least the vanishing of $\|M\|$ imply the vanishing of $\mathbf{|} M \mathbf{|}$?*

5.1 Properties of the integral foliated simplicial volume

1. The integral foliated simplicial volume is a homotopy invariant.

2. If $G := \pi_1(M)$ and X is a standard G-probability space, then the inclusion

$$C_*(M, \mathbf{Z}) = \mathbf{Z} \otimes_{\mathbf{Z}G} C_*(\widetilde{M}, \mathbf{Z}) \subset L^\infty(X, \mathbf{Z}) \otimes_{\mathbf{Z}G} C_*(\widetilde{M}, \mathbf{Z})$$

of chain complexes shows that $\lvert M \rvert^X \leq \|M\|_{\mathbf{Z}}$. Conversely, using integration $L^\infty(X, \mathbf{Z}) \longrightarrow \mathbf{R}$ of the coefficients we obtain that $\|M\| \leq \lvert M \rvert^X$.

3. If M is a simply connected oriented closed connected manifold, then $\lvert M \rvert = \|M\|_{\mathbf{Z}}$ [12]. More general, if the fundamental group of M is finite, then

$$\lvert M \rvert = \frac{1}{|\pi_1(M)|} \cdot \|M\|_{\mathbf{Z}}.$$

In particular, the integral foliated simplicial volume and the simplicial volume in general do not coincide.

4. Even worse: Let M be an oriented closed connected manifold dimension $n \geq 4$ and let $k \in \mathbf{N}$. Then Theorem (5.2) can be used to show the following [8]: There exists an n-manifold M_k admitting a continuous map $f_k \colon M_k \longrightarrow M$ such that $\pi_1(f_k) \colon \pi_1(M_k) \longrightarrow \pi_1(M)$ is an isomorphism and satisfying

$$\lvert M_k \rvert \geq \|M_k\| + k = \|M\| + k.$$

5. If there is a d-sheeted covering $N \longrightarrow M$ of oriented closed connected manifolds, then [8]

$$\lvert M \rvert^{G \times_H Y} = \frac{1}{d} \cdot \lvert N \rvert^{Y},$$
$$\lvert N \rvert^{\operatorname{res}^G_H X} \leq \lvert M \rvert^{X} \cdot d$$

holds for all standard G-spaces X and all standard H-spaces Y, where $G := \pi_1(M)$ and $H := \pi_1(N)$. Thus,

$$\lvert N \rvert = d \cdot \lvert M \rvert.$$

In particular, the integral foliated simplicial volume of flat manifolds vanishes [8].

Moreover, multiplicativity under finite coverings shows that $\lvert M \rvert = \|M\|$ *holds for all aspherical oriented closed connected surfaces* M [8].

6. By taking countable products of parameter spaces, one can show that the infimum over X in the definition of $\lvert M \rvert$ indeed is a minimum [8].

Using the ergodic decomposition theorem it is possible to restrict the infimum in the definition of $\lvert M \rvert$ to the class of ergodic parameter spaces [8].

7. The integral foliated simplicial volume is compatible with products in the following sense: If M and N are oriented closed connected manifolds, then
$$|M \times N| \leq c(\dim M + \dim N) \cdot |M| \cdot |N| .$$

8. Sauer recently proved that a certain invariant related to the integral foliated simplicial volume vanishes if the underlying oriented closed connected manifold is aspherical and has amenable fundamental group [13].

   Moreover, he proves (based on ideas of Gromov [6; p. 300f]) that this invariant is (up to a constant depending only on the dimension) for aspherical oriented closed connected manifolds bounded from above by the minimal volume [13].

   From these results one can also deduce the corresponding results for L^2-Betti numbers, because Sauer's invariant satisfies an estimate corresponding to Theorem (5.2).

To give an impression how the presence of a measure space can be used to "divide" cycles, we provide a sketch proof of 5.:

Proof. (of 5.) Let $p\colon N \longrightarrow M$ be a covering of oriented closed connected n-manifolds. Thus $d := |\deg p|$, the number of sheets of p, must be finite.

We first explain the proof of the corresponding equality $\|N\| = d \cdot \|M\|$ for the ordinary simplicial volume: The chain map
$$\frac{1}{d} \cdot C_*(p)\colon C_n(N, \mathbf{R}) \longrightarrow C_n(M, \mathbf{R})$$
shows that $\|M\| \leq 1/d \cdot \|N\|$. Conversely, the transfer
$$\begin{aligned} C_*(M, \mathbf{R}) &\longrightarrow C_*(N, \mathbf{R}) \\ \sigma &\longrightarrow \sum_{\substack{\tau \in \operatorname{map}(\Delta^*, N) \\ p \circ \tau = \sigma}} \tau \end{aligned}$$
maps fundamental cycles of M to fundamental cycles of N and is of norm at most d. Hence, $\|N\| \leq d \cdot \|M\|$.

Similarly, in the equivariant setting, we make use of the chain isomorphism
$$\begin{aligned} L^\infty(Y, \mathbf{Z}) \otimes_{\mathbf{Z}H} C_*(\widetilde{N}, \mathbf{Z}) &\cong L^\infty(Y, \mathbf{Z}) \otimes_{\mathbf{Z}H} \mathbf{Z}G \otimes_{\mathbf{Z}G} C_*(\widetilde{M}, \mathbf{Z}) \\ &\cong L^\infty(G \times_H Y, \mathbf{Z}) \otimes_{\mathbf{Z}G} C_*(\widetilde{M}, \mathbf{Z}) \end{aligned}$$
to show that $|M|^{G \times_H Y} = 1/d \cdot |N|^{Y}$. In order to prove the reverse inequality $|N|^{\operatorname{res}^G_H X} \leq d \cdot |M|^{X}$, we choose a lift $\widetilde{p}\colon \widetilde{N} \longrightarrow \widetilde{M}$ of p and consider the chain map
$$L^\infty(X, \mathbf{Z}) \otimes_{\mathbf{Z}G} C_*(\widetilde{M}, \mathbf{Z}) \longrightarrow L^\infty(\operatorname{res}^G_H X, \mathbf{Z}) \otimes_{\mathbf{Z}H} C_*(\widetilde{N}, \mathbf{Z})$$
that is given by summing up all $\widetilde{p}$-lifts of a singular simplex of $\widetilde{M}$ to $\widetilde{N}$. The effect of those chain maps when thinking of parametrised fundamental cycles as triangulations is illustrated in Figure (5.4). □

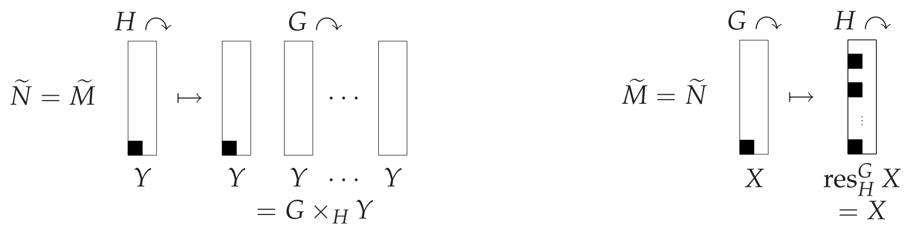

Figure (5.4): Multiplicativity of the integral foliated simplicial volume under finite coverings

5.2 Future work

- It would be nice to compute the integral foliated simplicial volume for more examples, such as S^1-principal bundles and hyperbolic manifolds.

- The main tool for computing the simplicial volume is bounded cohomology. It would be interesting to come up with a similar machinery for the integral foliated simplicial volume of aspherical oriented closed connected manifolds. This would possibly lead to a proportionality principle for the integral foliated simplicial volume and to a vanishing result for aspherical oriented closed connected manifolds with amenable fundamental group.

- It is not clear yet, whether the value $|M|^X$ does depend on the choice of the parameter space X.

- At least for aspherical oriented closed connected manifolds with residually finite fundamental group it should be possible to compute the integral foliated simplicial volume in terms of the integral simplicial volume of the finite coverings.

- Moreover, it would be interesting to show that the integral foliated simplicial volume of aspherical oriented closed connected manifolds with a selfmap of non-trivial degree is zero.

- It is clear that the obvious sum formula for the integral foliated simplicial volume of connected sums cannot hold (because the integral foliated simplicial volume is non-zero if the Euler characteristic is non-zero). It would be interesting to calculate the error term.

- Unfortunately, the class of aspherical oriented closed connected manifolds whose simplicial volume is known to be zero is quite small. It would be interesting to find more examples, in particular, it would be interesting to calculate the (integral foliated) simplicial volume of Davis's exotic aspherical manifolds.

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