

ISOMORPHISMS IN ℓ^1 -HOMOLOGY

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ABSTRACT. Taking the ℓ^1 -completion and the topological dual of the singular chain complex gives rise to ℓ^1 -homology and bounded cohomology respectively. In contrast to ℓ^1 -homology, major structural properties of bounded cohomology are well understood by the work of Gromov and Ivanov.

Based on an observation by Matsumoto and Morita, we derive a mechanism linking isomorphisms on the level of homology of Banach chain complexes to isomorphisms on the level of cohomology of the dual Banach cochain complexes and vice versa. Therefore, certain results on bounded cohomology can be transferred to ℓ^1 -homology. For example, we obtain a new proof of the fact that ℓ^1 -homology depends only on the fundamental group and that ℓ^1 -homology with twisted coefficients admits a description in terms of projective resolutions. The latter one in particular fills a gap in Park's approach.

In the second part, we demonstrate how ℓ^1 -homology can be used to get a better understanding of simplicial volume of non-compact manifolds.

1. INTRODUCTION

Semi-norms on singular homology contain valuable geometric information – the fundamental example of a topological invariant created this way is the simplicial volume of oriented, closed, connected manifolds, which is the ℓ^1 -semi-norm of the \mathbf{R} -fundamental class. However, singular homology itself is not an adequate algebraic tool for the study of the ℓ^1 -semi-norm. Only by passing to related theories such as bounded cohomology or ℓ^1 -homology the bigger picture becomes visible.

In contrast to ℓ^1 -homology, major structural properties of bounded cohomology are well understood by the work of Gromov [7] and Ivanov [9]. For example, bounded cohomology depends only on the fundamental group of the space in question [7, 9; p. 40, Theorem 4.3], bounded cohomology cannot see amenable normal subgroups of the fundamental group [7, 9; p. 40, Theorem 4.3], and bounded cohomology of spaces admits a description in terms of a certain flavour of homological algebra [9].

Matsumoto and Morita observed that ℓ^1 -homology of a space is trivial if and only if its bounded cohomology is trivial [17; Corollary 2.4]. Subsequently, they raised the natural question whether also ℓ^1 -homology depends only on the fundamental group. More generally one can ask how bounded cohomology and ℓ^1 -homology are related and whether there is some kind of duality. In the present article, we investigate to what extent such a duality holds.

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A convenient framework for this problem is the language of normed and Banach chain complexes, i.e., chain complexes of (complete) normed vector spaces whose boundary operators are bounded operators. Unlike taking algebraic duals of \mathbf{R} -chain complexes, taking topological duals of Banach chain complexes fails to commute with homology (Section 4.1). However, by exploiting the power of mapping cones, we prove in Section 4.2 the following replacement for the universal coefficient theorem:

Theorem (1.1) (Translation principle). *Let $f: C \rightarrow D$ be a morphism of Banach chain complexes and let $f': D' \rightarrow C'$ be its dual.*

- (1) *Then the induced homomorphism $H_*(f): H_*(C) \rightarrow H_*(D)$ is an isomorphism of vector spaces if and only if $H^*(f'): H^*(D') \rightarrow H^*(C')$ is an isomorphism of vector spaces.*
- (2) *Furthermore, if $H^*(f'): H^*(D') \rightarrow H^*(C')$ is an isometric isomorphism, then also $H_*(f): H_*(C) \rightarrow H_*(D)$ is an isometric isomorphism.*

In this article, the main examples for Banach (co)chain complexes are the ℓ^1 -chain complexes and bounded cochain complexes of spaces and of discrete groups respectively: The ℓ^1 -chain complex $C_*^{\ell^1}(X)$ of a topological space X is the ℓ^1 -completion of the singular chain complex of X with \mathbf{R} -coefficients and ℓ^1 -homology of X is defined to be the homology of this chain complex; dually, the bounded cochain complex $C_b^*(X)$ of X is the topological dual of $C_*^{\ell^1}(X)$ and bounded cohomology of X is defined to be the cohomology of $C_b^*(X)$. Similarly, the ℓ^1 -chain complex $C_*^{\ell^1}(G)$ of a discrete group G is obtained by taking the ℓ^1 -completion of the bar resolution, and the bounded cochain complex of G is the topological dual of $C_*^{\ell^1}(G)$.

Applying the translation principle to suitable chain maps in the realm of ℓ^1 -homology enables us to transfer many results concerning bounded cohomology to ℓ^1 -homology. In particular, this strategy provides a uniform, lightweight approach to the following results:

- Like bounded cohomology, ℓ^1 -homology of countable, connected CW-complexes depends only on the fundamental group and amenable normal subgroups of the fundamental group are a blind spot of ℓ^1 -homology (Corollary (5.2)).
- There is a characterisation of amenability of discrete groups through ℓ^1 -homology (Corollary (5.5)).
- There is a description of ℓ^1 -homology of spaces in terms of homological algebra; namely, ℓ^1 -homology of connected, countable CW-complexes coincides with ℓ^1 -homology of the fundamental group, and hence ℓ^1 -homology of such spaces can be computed via certain strong relatively projective resolutions (Corollary (5.8)).

Bouarich gave the first proof that ℓ^1 -homology depends only on the fundamental group [2; Corollaire 6]. His proof is based on the observation by Matsumoto and Morita, the fact that bounded cohomology of simply connected spaces vanishes, and an ℓ^1 -version of Brown's theorem. Moreover, Park [21; Corollary 4.2] already claimed that Corollary (5.2) holds. However, due to a gap in her argument, her proof is not complete. This issue is addressed in Caveats (5.7) and (5.9), which also show that it is not possible to imitate Ivanov's arguments in bounded cohomology in the setting of ℓ^1 -homology.

The results listed above might give the impression that ℓ^1 -homology is merely a shadow of bounded cohomology. However, there are also genuine applications of ℓ^1 -homology: For example, the simplicial volume of non-compact manifolds is not finite in general – it can even then be infinite if the manifold in question is the interior of a compact manifold with boundary. In this case, ℓ^1 -homology gives rise to a necessary and sufficient finiteness condition (Theorem (6.4)), which cannot be phrased in terms of bounded cohomology.

Organisation of this article. In Section 2, we introduce normed and Banach chain complexes. In Section 3, we review the basic definitions of ℓ^1 -homology and bounded cohomology of topological spaces as well as of discrete groups. Duality in the category of normed chain complexes and the proof of the translation principle are the topic of Section 4. In Section 5, we apply the translation principle to ℓ^1 -homology and we derive the consequences listed above. Finally, in Section 6, we demonstrate how to utilise ℓ^1 -homology to study the simplicial volume of non-compact manifolds.

2. HOMOLOGY OF NORMED CHAIN COMPLEXES

In this section, we introduce the basic objects of study – normed chain complexes and their homology. The main examples of these concepts are ℓ^1 -homology and bounded cohomology, which are reviewed in Section 3.

2.1. Normed and Banach chain complexes. Normed chain complexes are nothing but chain complexes in the category of normed vector spaces:

Definition (2.1) (Normed chain complexes).

- A *normed (co)chain complex* is a (co)chain complex (indexed over \mathbf{N}) consisting of normed real vector spaces, where all (co)boundary morphisms are bounded linear operators.
- A *Banach (co)chain complex* is a normed (co)chain complex consisting of Banach spaces.
- A *morphism of normed (co)chain complexes* is a (co)chain map between normed (co)chain complexes consisting of bounded linear operators. \diamond

Fundamental examples of normed chain complexes are the singular chain complex with real coefficients and the bar resolution of a discrete group with real coefficients (Section 3).

Definition (2.2) (Normed chain complexes – basic constructions). Let (C, ∂) be a normed chain complex.

- Because the boundary operator ∂ is bounded in each degree, it can be extended to a bounded boundary operator on the completion \overline{C} of C . The resulting Banach chain complex, denoted by $(\overline{C}, \overline{\partial})$, is called the *completion of C* .
- The *dual of C* is the Banach cochain complex (C', ∂') defined by

$$(C')^n := (C_n)',$$

where \cdot' stands for the topological dual vector space, together with the norm given by $\|f\|_\infty := \sup\{|f(c)| \mid c \in C_n, \|c\| = 1\}$ for all $f \in (C')^n$ and the coboundary operators

$$\begin{aligned} (\partial')^n &:= (\partial_{n+1})': (C')^n \longrightarrow (C')^{n+1} \\ f &\longmapsto (c \mapsto f(\partial_{n+1}(c))). \end{aligned} \quad \diamond$$

Clearly, if C is a normed chain complex, then $C' = (\overline{C})'$.

2.2. The induced semi-norm on homology. The presence of chain complexes calls for the investigation of the corresponding homology. In the case of normed chain complexes, the homology groups carry additional information – the induced semi-norm; for example, the simplicial volume is a topological invariant defined in terms of such a semi-norm (Section 3.1.3).

Definition (2.3) (Semi-norm on homology). Let (C, ∂) be a normed chain complex, and let $n \in \mathbf{N}$. The norm $\|\cdot\|$ on C_n induces a semi-norm on the n -th homology group $H_n(C) := \ker \partial_n / \text{im } \partial_{n+1}$ as follows: If $\alpha \in H_n(C)$, then

$$\|\alpha\| := \inf\{\|c\| \mid c \in C_n, \partial_n(c) = 0, [c] = \alpha\}. \quad \diamond$$

Of course, the same definition applies also to normed cochain complexes.

Because the images of the boundary operators of a normed chain complex are not necessarily closed, the induced semi-norm on homology in general is not a norm; this can even happen if the underlying normed (co)chain complex is Banach [25, 26].

Despite of the fact that the homology of a normed chain complex and the homology of the corresponding completion in general are quite different, the semi-norms are related. In fact, in order to understand the semi-norms on the homology of normed chain complexes, it suffices to consider the case of Banach chain complexes, which is shown by approximating boundaries [24, 13; Lemma 2.9, Proposition 1.7]:

Proposition (2.4). *Let D be a normed chain complex and let C be a dense subcomplex. Then the map $H_*(C) \longrightarrow H_*(D)$ induced by the inclusion is isometric.*

Moreover, one can also compute the induced semi-norm on $H_*(C)$ via the semi-norm on $H^*(C')$ (Theorem (4.4)).

3. ℓ^1 -HOMOLOGY AND BOUNDED COHOMOLOGY

Taking the completion and the topological dual of the singular chain complex with respect to the ℓ^1 -norm gives rise to ℓ^1 -homology and bounded cohomology respectively (Section 3.1). Also the bar resolution of a discrete group admits an ℓ^1 -norm – leading to ℓ^1 -homology and bounded cohomology of discrete groups (Section 3.2). Both constructions can be decorated with equivariant Banach modules, which yields the corresponding theories with (twisted) coefficients (Sections 3.2 and 3.3).

3.1. ℓ^1 -Homology and bounded cohomology of spaces. We start with the key example of a normed chain complex:

Definition (3.1) (ℓ^1 -Norm on the singular chain complex). Let (X, A) be a pair of topological spaces.

- The ℓ^1 -norm on the singular chain complex $C_*(X)$ with real coefficients is defined as follows: For a chain $c = \sum_{j=0}^k a_j \cdot \sigma_j \in C_n(X)$ in reduced form we set

$$\|c\|_1 := \sum_{j=0}^k |a_j|.$$

- The induced semi-norm on the quotient $C_*(X, A) = C_*(X)/C_*(A)$ is a norm because the subcomplex $C_*(A)$ is ℓ^1 -closed in $C_*(X)$; this norm on $C_*(X, A)$ is also denoted by $\|\cdot\|_1$. \diamond

The boundary operator $\partial_n: C_n(X, A) \rightarrow C_{n-1}(X, A)$ is a bounded operator with respect to the ℓ^1 -norm of operator norm at most $(n+1)$. Hence, $C_*(X, A)$ is a normed chain complex. Clearly, $C_*(X)$ and $C_*(X, A)$ are in general not complete and thus these complexes are no Banach chain complexes.

On the other hand, for $p \in (1, \infty]$, the singular chain complex equipped with the ℓ^p -norm is in general *not* a normed chain complex in the sense of Definition (2.1) [13; Proposition 2.11].

Definition (3.2) (ℓ^1 -Homology and bounded cohomology of spaces). Let (X, A) be a pair of topological spaces.

- The ℓ^1 -chain complex of (X, A) is the completion $C_*^{\ell^1}(X, A)$ of the normed chain complex $C_*(X, A)$ with respect to $\|\cdot\|_1$. We abbreviate $C_*^{\ell^1}(X, \emptyset)$ by $C_*^{\ell^1}(X)$.
- Then ℓ^1 -homology of (X, A) is defined as

$$H_*^{\ell^1}(X, A) := H_*(C_*^{\ell^1}(X, A)).$$

- Dually, the bounded cochain complex of (X, A) is the dual $C_b^*(X, A)$ of the normed chain complex $C_*(X, A)$.
- Bounded cohomology of (X, A) is given by

$$H_b^*(X, A) := H^*(C_b^*(X, A)).$$

- The semi-norms on ℓ^1 -homology and bounded cohomology induced by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively are also denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$. \diamond

3.1.1. Functoriality. If $f: (X, A) \rightarrow (Y, B)$ is a continuous map of pairs of topological spaces, then the induced map $C_*(f): C_*(X, A) \rightarrow C_*(Y, B)$ is a morphism of normed chain complexes. Consequently, we obtain induced morphisms $C_*^{\ell^1}(f)$ and $C_b^*(f)$, as well as maps $H_*^{\ell^1}(f): H_*^{\ell^1}(X, A) \rightarrow H_*^{\ell^1}(Y, B)$ and $H_b^*(f): H_b^*(X, A) \rightarrow H_b^*(Y, B)$; clearly, both $H_*^{\ell^1}(\cdot)$ and $H_b^*(\cdot)$ are functorial with respect to composition.

3.1.2. *Basic properties.* Standard arguments show that ℓ^1 -homology and bounded cohomology are homotopy invariant and admit a long exact sequence for pairs of topological spaces [13; Proposition 2.7]. Using self-maps of the circle of non-trivial degree one finds that $H_1^{\ell^1}(X) = 0$ and $H_b^1(X) = 0$ holds for all spaces X [17, 13; Corollary 2.7, Proposition 2.7]. However, both ℓ^1 -homology and bounded cohomology do *not* satisfy excision [3, 18]. This failure of excision is both a curse and a blessing. On the one hand, the lack of excision makes concrete computations via the usual divide and conquer approach significantly harder; on the other hand, it turns out that both bounded cohomology and ℓ^1 -homology depend only on the fundamental group and hence can be computed in terms of certain nice resolutions (Corollary (5.2) and Corollary (5.8)).

3.1.3. *Simplicial volume.* An example of valuable geometric information encoded in a semi-norm on homology is the simplicial volume introduced by Gromov [7]. The simplicial volume is a homotopy invariant linked to Riemannian geometry in various ways and can be viewed as a topological approximation of the Riemannian volume [7].

Definition (3.3). Let M be an oriented, closed, connected n -manifold with \mathbf{R} -fundamental class $[M] \in H_n(M)$. Then the *simplicial volume of M* is defined as

$$\|M\| := \|[M]\|_1 = \inf \{ \|c\|_1 \mid c \in C_n(M) \text{ is an } \mathbf{R}\text{-fundamental cycle of } M \}. \quad \diamond$$

Using self-maps of non-trivial degree one sees that the simplicial volume of spheres and tori is zero. On the other hand, straightening simplices to geodesic simplices shows that the simplicial volume of closed hyperbolic manifolds is non-zero [28, 8].

However, it is in general very difficult to compute the simplicial volume by geometric means. In view of Proposition (2.4) and Theorem (4.4) below, it is possible to use ℓ^1 -homology and bounded cohomology to compute the simplicial volume. For example, this approach shows that the simplicial volume of all manifolds with amenable fundamental group is zero. Conversely, we can deduce that ℓ^1 -homology and bounded cohomology of closed hyperbolic manifolds are non-trivial.

3.2. **ℓ^1 -Homology and bounded cohomology of discrete groups.** For a discrete group G , we write $C_*(G)$ for the corresponding bar resolution with real coefficients; more explicitly, $C_n(G)$ is the free $\mathbf{R}G$ -module with basis $([g_1 | \dots | g_n])_{g \in G^n}$, and the boundary operator $C_n(G) \rightarrow C_{n-1}(G)$ is the G -linear map determined uniquely by

$$\begin{aligned} C_n(G) &\longrightarrow C_{n-1}(G) \\ [g_1 | \dots | g_n] &\longmapsto g_1 \cdot [g_2 | \dots | g_n] \\ &\quad + \sum_{j=1}^{n-1} (-1)^j \cdot [g_1 | \dots | g_{j-1} | g_j \cdot g_{j+1} | g_{j+2} | \dots | g_n] \\ &\quad + (-1)^n \cdot [g_1 | \dots | g_{n-1}]. \end{aligned}$$

Definition (3.4) (ℓ^1 -Norm on the bar resolution of discrete groups). Let G be a discrete group, and let $n \in \mathbf{N}$. For $c = \sum_{g \in G^{n+1}} a_g \cdot g_0 \cdot [g_1 | \dots | g_n] \in C_n(G)$ we define

$$\|c\|_1 := \sum_{g \in G^{n+1}} |a_g|. \quad \diamond$$

The group G acts isometrically on $C_*(G)$ and $C_*(G)$ is a normed chain complex with respect to the ℓ^1 -norm; in particular, we obtain the corresponding completions and topological duals:

Definition (3.5) (ℓ^1 -chains and bounded cochains of discrete groups). Let G be a discrete group.

- The ℓ^1 -chain complex of G is the completion $C_*^{\ell^1}(G)$ of the normed chain complex $C_*(G)$ with respect to $\|\cdot\|_1$.
- The bounded cochain complex of G is the dual $C_b^*(G)$ of the normed chain complex $C_*(G)$. \diamond

In order to define ℓ^1 -homology and bounded cohomology of discrete groups (with coefficients), we need some terminology from the category of Banach G -modules: A *Banach G -module* is a Banach space equipped with an isometric (left) G -action. If U and V are two Banach G -modules, then the projective tensor product $U \overline{\otimes} V$ and the space $B(U, V)$ of bounded linear functions from U to V are Banach G -modules with respect to the following, diagonal, G -actions: For all $g \in G$ one sets

$$\begin{aligned} \forall_{u \in U} \quad \forall_{v \in V} \quad g \cdot (u \otimes v) &:= (g \cdot u) \otimes (g \cdot v), \text{ and} \\ \forall_{f \in B(U, V)} \quad g \cdot f &:= (u \mapsto g \cdot f(g^{-1} \cdot u)). \end{aligned}$$

For a Banach G -module V the set of *invariants* of V is defined by

$$V^G := \{v \in V \mid \forall_{g \in G} \quad g \cdot v = v\};$$

the set of *coinvariants* of V is the quotient $V_G := V/\overline{W}$, where $W \subset V$ is the subspace generated by the set $\{g \cdot v - v \mid v \in V, g \in G\}$. It is not difficult to see that there is an isometric isomorphism $(V_G)' \cong (V')^G$, which can be viewed as an adjointness property.

A *Banach G -(co)chain complex* is a normed (co)chain complex consisting of Banach G -modules whose (co)boundary operators are G -equivariant. For example, $C_*^{\ell^1}(G)$ is a Banach G -chain complex. A *morphism of Banach G -(co)chain complexes* is just a morphism of normed (co)chain complexes that is G -equivariant. Notions such as the invariants etc. have obvious analogues on the level of Banach G -(co)chain complexes.

Now the definition of ℓ^1 -homology and bounded cohomology of discrete groups is a straightforward adaption of the definition of group (co)homology in terms of the bar resolution:

Definition (3.6) (ℓ^1 -Homology and bounded cohomology of discrete groups). Let G be a discrete group, and let V be a Banach G -module.

- We write

$$C_*^{\ell^1}(G; V) := C_*^{\ell^1}(G) \overline{\otimes} V \quad \text{and} \quad C_b^*(G; V) := B(C_*^{\ell^1}(G), V).$$

- The ℓ^1 -homology of G with coefficients in V , denoted by $H_*^{\ell^1}(G; V)$, is the homology of the Banach chain complex $C_*^{\ell^1}(G; V)_G$.
- Bounded cohomology of G with coefficients in V , denoted by $H_b^*(G; V)$, is the cohomology of the Banach cochain complex $C_b^*(G; V)^G$. \diamond

Notice that $C_b^*(G; V')$ is isometrically G -isomorphic to $(C_*^{\ell^1}(G; V))'$; in particular, we have $C_b^*(G; \mathbf{R}) = C_b^*(G)$, where \mathbf{R} is equipped with the trivial G -action. For brevity, we write $H_*^{\ell^1}(G) := H_*^{\ell^1}(G; \mathbf{R})$ and $H_b^*(G) := H_b^*(G; \mathbf{R})$.

Moreover, the ℓ^1 -norm on $C_*^{\ell^1}(G)$ and the norm on V induce norms on $C_*^{\ell^1}(G; V)$ and $C_b^*(G; V)$, and hence they induce semi-norms on $H_*^{\ell^1}(G; V)$ and $H_b^*(G; V)$. These semi-norms are also denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively.

3.2.1. *ℓ^1 -Homology and bounded cohomology in degree 0.* Almost the same calculations as in ordinary group (co)homology show that

$$H_0^{\ell^1}(G; V) \cong V_G \quad \text{and} \quad H_b^0(G; V) \cong V^G$$

holds for all discrete groups G and all Banach G -modules V .

3.2.2. *Functoriality.* Let $\varphi: G \rightarrow H$ be a homomorphism of discrete groups, let V be a Banach G -module and let W be a Banach H -module. Then

$$\begin{aligned} C_n^{\ell^1}(\varphi): C_n^{\ell^1}(G) &\longrightarrow \varphi^*(C_n^{\ell^1}(H)) \\ g_0 \cdot [g_1 | \dots | g_n] &\longmapsto \varphi(g_0) \cdot [\varphi(g_1) | \dots | \varphi(g_n)] \end{aligned}$$

defines a morphism $C_*^{\ell^1}(\varphi): C_*^{\ell^1}(G) \rightarrow \varphi^*C_*^{\ell^1}(H)$ of Banach G -chain complexes of norm 1; here, $\varphi^*(\cdot)$ stands for the Banach G -module structure on the Banach H -module in question that is induced by φ . In particular, for any morphism $f: V \rightarrow \varphi^*W$ of Banach G -modules, the map

$$C_*^{\ell^1}(\varphi; f) := C_*^{\ell^1}(\varphi) \overline{\otimes} f: C_*^{\ell^1}(G; V) \longrightarrow \varphi^*(C_*^{\ell^1}(H; W))$$

is a morphism of Banach G -chain complexes (of norm at most $\|f\|$). Analogously, for any morphism $f: \varphi^*W \rightarrow V$ of Banach G -modules,

$$C_b^*(\varphi; f) := B(C_*^{\ell^1}(\varphi), f): \varphi^*(C_b^*(H; W)) \longrightarrow C_b^*(G; V)$$

is a morphism of Banach G -cochain complexes (of norm at most $\|f\|$).

Let $p: (\varphi^*C_b^{\ell^1}(H; W))_G \rightarrow C_b^{\ell^1}(H; W)_H$ and $i: C_b^*(H; W)^H \rightarrow (\varphi^*C_b^*(H; W))^G$ denote the canonical projection and the inclusion respectively. Then we write

$$\begin{aligned} H_*^{\ell^1}(\varphi; f) &:= H_*(p \circ C_*^{\ell^1}(\varphi; f)_G): H_*^{\ell^1}(G; V) \longrightarrow H_*^{\ell^1}(H; W), \\ H_b^*(\varphi; f) &:= H^*(C_b^*(\varphi; f)^G \circ i): H_b^*(H; W) \longrightarrow H_b^*(G; V). \end{aligned}$$

3.2.3. *Strong relatively injective/relatively projective resolutions.* Both ℓ^1 -homology and bounded cohomology of discrete groups enjoy the same flexibility as ordinary group (co)homology: namely, both theories can be computed by means of relative homological algebra as studied by Brooks, Ivanov, Monod, and Park [3, 9, 19, 21].

As in the classical case, there is a distinguished class of resolutions – so-called strong relatively projective resolutions and strong relatively injective resolutions – and a corresponding fundamental lemma of homological algebra granting existence and uniqueness of certain morphisms of Banach G -chain complexes [13; Appendix A]; for example, the Banach (co)chain complexes $C_*^{\ell^1}(G; V)$ and $C_b^*(G; V)$ together with the obvious augmentation maps are strong relatively projective/injective G -resolutions of V [13; Proposition 2.19]. Therefore, we obtain [13; Theorem 2.18]:

Theorem (3.7). *Let G be a discrete group and let V be a Banach G -module.*

- (1) *For any strong relatively projective G -resolution $(C, \eta: C_0 \rightarrow V)$ of V there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*

$$H_*^{\ell^1}(G; V) \cong H_*(C_G).$$

- (2) *For any strong relatively injective G -resolution $(C, \eta: V \rightarrow C^0)$ of V there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*

$$H_b^*(G; V) \cong H^*(C^G).$$

- (3) *If $(C, \eta: C_0 \rightarrow \mathbf{R})$ is a strong relatively projective G -resolution of the trivial Banach G -module \mathbf{R} , then there are canonical isomorphisms (degreewise isomorphisms of semi-normed vector spaces)*

$$\begin{aligned} H_*^{\ell^1}(G; V) &\cong H_*((C \bar{\otimes} V)_G), \\ H_b^*(G; V) &\cong H^*(B(C, V)^G). \end{aligned}$$

The semi-norms on $H_*^{\ell^1}(\cdot; \cdot)$ and $H_b^*(\cdot; \cdot)$ induced by the bar resolutions $C_*^{\ell^1}(\cdot; \cdot)$ and $C_b^*(\cdot; \cdot)$ coincide with the canonical semi-norms in the sense of Ivanov [9, 21, 19; Corollary 3.6.1, Corollary 2.3, Corollary 7.4.7]. However, it is clear that not every strong relatively projective/injective resolution induces the same semi-norm in (co)homology.

Bühler developed a description of ℓ^1 -homology and bounded cohomology as derived functors via exact categories [5], thereby providing an even more conceptual approach.

3.3. ℓ^1 -Homology and bounded cohomology of spaces with twisted coefficients. Similarly to singular homology and singular cohomology there are also versions of ℓ^1 -homology and bounded cohomology of spaces with twisted coefficients:

Definition (3.8) (ℓ^1 -Homology and bounded cohomology with twisted coefficients). Let X be a connected topological space with fundamental group G that admits a universal covering space \tilde{X} , and let V be a Banach G -module.

- The ℓ^1 -chain complex of X with twisted coefficients in V is defined as the Banach chain complex of coinvariants

$$C_*^{\ell^1}(X; V) := (C_*^{\ell^1}(\tilde{X}) \bar{\otimes} V)_G.$$

Here, $C_*^{\ell^1}(\tilde{X})$ inherits the G -action from the action of the fundamental group on the universal covering \tilde{X} .

- The ℓ^1 -homology of X with twisted coefficients in V , denoted by $H_*^{\ell^1}(X; V)$, is the homology of the Banach chain complex $C_*^{\ell^1}(X; V)$.
- The bounded cochain complex of X with twisted coefficients in V is defined as the Banach cochain complex of invariants

$$C_b^*(X; V) := B(C_*^{\ell^1}(\tilde{X}), V)^G.$$

- Bounded cohomology of X with twisted coefficients in V is the cohomology of the Banach cochain complex $C_b^*(X; V)$ and is denoted by $H_b^*(X; V)$. \diamond

The ℓ^1 -chain complex and the bounded cochain complex of X as defined in Definition (3.2) can be recovered from this definition by taking \mathbf{R} with the trivial G -action as coefficients [13; Proposition 2.23].

4. DUALITY

In this section, we investigate the relation induced by the evaluation map between homology of a normed chain complex and cohomology of its dual cochain complex. Unlike taking algebraic duals of \mathbf{R} -chain complexes, taking topological duals of normed chain complexes fails to commute with homology (Section 4.1). Section 4.2 is devoted to the proof of the translation principle (Theorem (1.1)), showing that it is still possible to transfer certain information from homology of a Banach chain complex to cohomology of the dual complex and vice versa.

4.1. Linking homology and cohomology. Evaluation links homology of a normed chain complex to cohomology of its dual cochain complex: If C is a normed chain complex and $n \in \mathbf{N}$, then the evaluation map $C'^n \otimes C_n \rightarrow \mathbf{R}$ induces a linear map

$$\langle \cdot, \cdot \rangle: H^n(C') \otimes H_n(C) \rightarrow \mathbf{R},$$

the so-called *Kronecker product*. Similarly, we obtain a map $\overline{H}^n(C') \rightarrow (\overline{H}_n(C))'$, where \overline{H} denotes reduced (co)homology, i.e., the kernel modulo the *closure* of the image of the (co)boundary operator.

Taking the algebraic dual is compatible with taking homology: For all \mathbf{R} -chain complexes C the map $H^*(\text{hom}_{\mathbf{R}}(C, \mathbf{R})) \rightarrow \text{hom}_{\mathbf{R}}(H_*(C), \mathbf{R})$ induced by evaluation is an isomorphism by the universal coefficient theorem. However, taking topological duals, even of complete normed chain complexes, fails to commute with taking homology:

Remark (4.1). *There is no obvious duality isomorphism between homology and cohomology of Banach chain complexes:*

Let C be a Banach chain complex. Then we have the following commutative diagram

$$\begin{array}{ccc} H^*(C') & \longrightarrow & \text{hom}_{\mathbf{R}}(H_*(C), \mathbf{R}) \\ \downarrow & \searrow & \uparrow \\ \overline{H}^*(C') & \longrightarrow & (\overline{H}_*(C))' \end{array}$$

where the horizontal arrows are the homomorphisms induced by the Kronecker products (i.e., they are induced by evaluation of elements in C' on elements in C), the left vertical arrow is the canonical projection and the right vertical arrow is the composition $(\overline{H}_*(C))' \hookrightarrow \text{hom}_{\mathbf{R}}(\overline{H}_*(C), \mathbf{R}) \hookrightarrow \text{hom}_{\mathbf{R}}(H_*(C), \mathbf{R})$ of inclusions.

The lower horizontal morphism, and hence also the diagonal morphism, is surjective by the Hahn-Banach theorem. Moreover, Matsumoto and Morita showed that the diagonal morphism is injective if and only if $H^*(C') = \overline{H}^*(C')$ holds [17; Theorem 2.3].

Obviously, this is not the case in general. It is even wrong if $C = C_*^{\ell^1}(X)$ for certain topological spaces X [25, 26]. Hence, there is no obvious duality between ℓ^1 -homology and bounded cohomology.

In addition, the lower horizontal arrow is in general not injective: The kernel of the evaluation map

$$\ker \partial'^{n+1} \longrightarrow (\ker \partial_n / \overline{\text{im } \partial_{n+1}})' = (\overline{H}_n(C))'$$

equals $({}^\perp \text{im}(\partial'^n))^\perp$, which is the weak*-closure of $\text{im } \partial'^n$ [23; Theorem 4.7]. Furthermore, the norm-closure $\overline{\text{im } \partial'^n}$ and the weak*-closure $({}^\perp \text{im}(\partial'^n))^\perp$ coincide if and only if $\text{im } \partial_{n+1}$ is closed [23; Theorem 4.14]. Thus there is also no obvious duality isomorphism between reduced ℓ^1 -homology and reduced bounded cohomology. \diamond

Nevertheless, the Kronecker product is strong enough to give sufficient conditions for (co)homology classes to be non-trivial. For example, if $\alpha \in H_*(C)$ and $\varphi \in H^*(C')$ with $\langle \varphi, \alpha \rangle = 1$, then neither α , nor φ can be zero. This effect can be used to show that ℓ^1 -homology and bounded cohomology of certain surface groups are non-trivial [18].

4.2. Transferring isomorphisms – proof of Theorem (1.1).

4.2.1. *Method of proof.* The proof of the translation principle (Theorem (1.1)) relies on the following three tools:

- (1) *Duality principle.* There is the following relation between homology of Banach chain complexes and cohomology of their duals, which has been discovered by Johnson as well as by Matsumoto and Morita [10, 17, 13; Proposition 1.2, Corollary 2.4, Theorem 3.5].

Theorem (4.2) (Duality principle). *Let C be a Banach chain complex. Then $H_*(C)$ vanishes if and only if $H^*(C')$ vanishes.*

Here, the “*” carries the meaning “All of the $H_n(C)$ are zero if and only if all of the $H^n(C')$ are zero.” The key to lifting this duality principle to morphisms of Banach chain complexes is to apply the duality principle to the mapping cone of the morphism in question.

- (2) *Mapping cones.* Mapping cones of chain maps are a device translating questions about isomorphisms on homology into questions about the vanishing of certain homology groups; the exact definition of mapping cones in the context of Banach chain complexes is given in Section 4.2.2 below.

Proposition (4.3). *Let $f: C \longrightarrow D$ be a morphism of normed chain complexes.*

- (a) *The induced map $H_*(f): H_*(C) \longrightarrow H_*(D)$ is an isomorphism of vector spaces if and only if $H_*(\text{Cone}(f)) = 0$. Of course, the analogous statement for morphisms of normed cochain complexes also holds.*
- (b) *There is a natural isomorphism*

$$\text{Cone}(f)' \cong \Sigma \text{Cone}(-f')$$

of normed cochain complexes, relating the mapping cones of f and $-f'$.

The suspension Σ just shifts the (co)chain complex in question by $+1$; notice that the boundary operators do not change their signs.

The first part of Proposition (4.3) is a classic fact from homological algebra [29; Section 1.5]; a straightforward calculation proves the second part.

- (3) *Duality principle for semi-norms.* The third ingredient for the proof of the translation principle is the following observation of Gromov [7, 1, 13; p. 17, Proposition F.2.2, Theorem 3.8], relating the semi-norm on homology to the semi-norm on cohomology of the dual.

Theorem (4.4) (Duality principle for semi-norms). *Let C be a normed chain complex and let $n \in \mathbf{N}$. Then*

$$\|\alpha\| = \sup \left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in H^n(C') \text{ and } \langle \varphi, \alpha \rangle = 1 \right\}$$

holds for each $\alpha \in H_n(C)$; here, $\sup \emptyset := 0$.

However, the semi-norm on cohomology of the dual can in general not be computed in terms of the semi-norm on homology, because it can happen that the reduced homology $\overline{H}_*(C)$ is zero while $\overline{H}^*(C')$ is non-zero (cf. Remark (4.1)).

4.2.2. *Mapping cones.* For the sake of completeness, we recall the definition of mapping cones of morphisms of Banach chain complexes:

Definition (4.5) (Mapping cones).

- Let $f: (C, \partial^C) \rightarrow (D, \partial^D)$ be a morphism of normed chain complexes. Then the *mapping cone* of f , denoted by $\text{Cone}(f)$, is the normed chain complex defined by

$$\text{Cone}(f)_n := C_{n-1} \oplus D_n,$$

linked by the boundary operator that is given by the matrix

$$\begin{array}{ccc} \text{Cone}(f)_n = C_{n-1} \oplus D_n & & \\ \begin{pmatrix} -\partial^C & 0 \\ f & \partial^D \end{pmatrix} \downarrow & \begin{array}{c} \searrow \\ -\partial^C \downarrow \\ \searrow \\ f \downarrow \\ \searrow \\ \partial^D \downarrow \end{array} & \\ \text{Cone}(f)_{n-1} = C_{n-2} \oplus D_{n-1} & & \end{array}$$

- Dually, if $f: (D, \delta_D) \rightarrow (C, \delta_C)$ is a morphism of normed cochain complexes, then the *mapping cone* of f , also denoted by $\text{Cone}(f)$, is the normed cochain complex defined by

$$\text{Cone}(f)^n := D^{n+1} \oplus C^n$$

with the coboundary operator determined by the matrix

$$\begin{array}{ccc} \text{Cone}(f)^n = D^{n+1} \oplus C^n & & \\ \begin{pmatrix} -\delta_D & 0 \\ f & \delta_C \end{pmatrix} \downarrow & \begin{array}{c} \searrow \\ -\delta_D \downarrow \\ \searrow \\ f \downarrow \\ \searrow \\ \delta_C \downarrow \end{array} & \\ \text{Cone}(f)^{n+1} = D^{n+2} \oplus C^{n+1} & & \end{array}$$

In the first case, we equip the mapping cone with the direct sum of the norms, in the second case, we use the maximum norm. \diamond

4.2.3. *Proof of the translation principle.* To prove the translation principle we just need to assemble the pieces collected in the previous paragraphs in the right way:

Proof (of Theorem (1.1)). The first part follows by fusing properties of mapping cones with the duality principle: The induced homomorphism $H_*(f)$ is an isomorphism if and only if $H_*(\text{Cone}(f)) = 0$. In view of the duality principle and the compatibility of mapping cones with taking the topological dual, this is equivalent to

$$0 = H^*(\text{Cone}(f)') \cong H^*(\Sigma\text{Cone}(-f')) = H^{*-1}(\text{Cone}(-f'));$$

the duality principle is applicable because the cone of a morphism of Banach chain complexes is a Banach chain complex. On the other hand, the $H^{*-1}(\text{Cone}(-f'))$ are all zero if and only if $H^*(-f'): H^*(D') \rightarrow H^*(C')$ is an isomorphism. Moreover, $H^*(f') = -H^*(-f')$, and therefore the first part is shown.

For the second part, it remains to prove that $H_*(f)$ is isometric whenever $H^*(f')$ is an isometric isomorphism. Let $n \in \mathbf{N}$ and let $\alpha \in H_n(C)$. Using the duality principle for semi-norms twice, we obtain

$$\begin{aligned} \|H_n(f)(\alpha)\| &= \sup \left\{ \frac{1}{\|\psi\|_\infty} \mid \psi \in H^n(D') \text{ and } \langle \psi, H_n(f)(\alpha) \rangle = 1 \right\} \\ &= \sup \left\{ \frac{1}{\|\psi\|_\infty} \mid \psi \in H^n(D') \text{ and } \langle H^n(f')(\psi), \alpha \rangle = 1 \right\} \\ &= \sup \left\{ \frac{1}{\|H^n(f')(\psi)\|_\infty} \mid \psi \in H^n(D') \text{ and } \langle H^n(f')(\psi), \alpha \rangle = 1 \right\} \\ &= \sup \left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in H^n(C') \text{ and } \langle \varphi, \alpha \rangle = 1 \right\} \\ &= \|\alpha\|. \end{aligned} \quad \square$$

5. ISOMORPHISMS IN ℓ^1 -HOMOLOGY

In this section, we apply the translation mechanism established in the previous section to ℓ^1 -homology, thereby gaining a uniform, lightweight approach to proving that ℓ^1 -homology depends only on the fundamental group (Section 5.1), that ℓ^1 -homology cannot see amenable, normal subgroups (Section 5.1 and 5.2) and that ℓ^1 -homology of spaces can be computed in terms of certain projective resolutions (Section 5.3).

5.1. **Isomorphisms in ℓ^1 -homology of spaces.** We start with the simplest applications of this type, concerning ℓ^1 -homology of spaces with \mathbf{R} -coefficients:

Corollary (5.1). *Let $f: (X, A) \rightarrow (Y, B)$ be a continuous map of pairs of topological spaces.*

- (1) *The induced homomorphism $H_*^{\ell^1}(f): H_*^{\ell^1}(X, A) \rightarrow H_*^{\ell^1}(Y, B)$ is an isomorphism if and only if $H_b^*(f): H_b^*(Y, B) \rightarrow H_b^*(X, A)$ is an isomorphism.*
- (2) *If $H_b^*(f): H_b^*(Y, B) \rightarrow H_b^*(X, A)$ is an isometric isomorphism, then $H_*^{\ell^1}(f)$ is also an isometric isomorphism.*
- (3) *In particular, $H_*^{\ell^1}(X, A)$ vanishes if and only if $H_b^*(X, A)$ vanishes.*

Proof. By definition, $C_b^*(X, A) = (C_*^{\ell^1}(X, A))'$ and $C_b^*(Y, B) = (C_*^{\ell^1}(Y, B))'$. The cochain map $C_b^*(f): C_b^*(Y, B) \rightarrow C_b^*(X, A)$ coincides with $(C_*^{\ell^1}(f))'$. Applying the translation principle Theorem (1.1) to $C_*^{\ell^1}(f)$ proves the Corollary. \square

A discrete group A is *amenable* if there is a left-invariant mean on the set $B(A, \mathbf{R})$ of bounded functions from A to \mathbf{R} , i.e., if there is a linear map $m: B(A, \mathbf{R}) \rightarrow \mathbf{R}$ satisfying

$$\forall_{f \in B(A, \mathbf{R})} \quad \forall_{a \in A} \quad m(f) = m(b \mapsto f(a^{-1} \cdot b))$$

and

$$\forall_{f \in B(A, \mathbf{R})} \quad \inf\{f(a) \mid a \in A\} \leq m(f) \leq \sup\{f(a) \mid a \in A\}.$$

For instance, all finite and all Abelian groups are amenable. Moreover, the class of amenable groups is closed under taking subgroups and quotients. An example of a non-amenable group is the free group $\mathbf{Z} * \mathbf{Z}$. A detailed discussion of amenability can be found in Paterson's book [22].

Corollary (5.2) (Mapping theorem for ℓ^1 -homology). *Let $f: X \rightarrow Y$ be a continuous map between connected, countable CW-complexes such that $\pi_1(f): \pi_1(X) \rightarrow \pi_1(Y)$ is surjective and has amenable kernel. Then the induced homomorphism*

$$H_*^{\ell^1}(f): H_*^{\ell^1}(X) \rightarrow H_*^{\ell^1}(Y)$$

is an isometric isomorphism.

Proof. It is a classical result in the theory of bounded cohomology that in this situation $H_b^*(f): H_b^*(Y) \rightarrow H_b^*(X)$ is an isometric isomorphism [7, 9; p. 40, Theorem 4.3]. Therefore, Corollary (5.1) completes the proof. \square

Applying the mapping theorem to the classifying map $X \rightarrow B\pi_1(X)$ shows in particular that the ℓ^1 -homology of a connected, countable CW-complex X depends only on the fundamental group.

5.2. Isomorphisms in ℓ^1 -homology of discrete groups. For ℓ^1 -homology of discrete groups the translation principle takes the following form:

Corollary (5.3). *Let $\varphi: G \rightarrow H$ be a homomorphism of discrete groups, let V be a Banach G -module, let W be a Banach H -module and suppose that $f: V \rightarrow \varphi^*W$ is a morphism of Banach G -modules.*

- (1) *Then the homomorphism $H_*^{\ell^1}(\varphi; f): H_*^{\ell^1}(G; V) \rightarrow H_*^{\ell^1}(H; W)$ is an isomorphism if and only if $H_b^*(\varphi; f'): H_b^*(H; W') \rightarrow H_b^*(G; V')$ is an isomorphism.*
- (2) *If $H_b^*(\varphi; f')$ is an isometric isomorphism, then so is $H_*^{\ell^1}(\varphi; f)$.*
- (3) *In particular, $H_*^{\ell^1}(G; V) \cong H_*^{\ell^1}(1; V_G)$ if and only if $H_b^*(G; V') \cong H_b^*(1; (V')^G)$.*

Proof. By definition, we have

$$\begin{aligned} H_*^{\ell^1}(\varphi; f) &= H_*(p \circ C_*^{\ell^1}(\varphi; f)_G), \\ H_b^*(\varphi; f') &= H^*(C_b^*(\varphi; f')^G \circ i), \end{aligned}$$

where $p: (\varphi^*C_*^{\ell^1}(H; W))_G \rightarrow C_*^{\ell^1}(H; W)_H$ and $i: C_b^*(H; W')^H \rightarrow (\varphi^*C_b^*(H; W'))^G$ denote the canonical projection and the inclusion respectively.

$$\begin{array}{ccccc}
(C_*^{\ell^1}(H; W)_H)' & \xlongequal{\quad} & (C_*^{\ell^1}(H; W)')^H & \xlongequal{\quad} & C_b^*(H; W')^H \\
\downarrow p' & & & & \downarrow i \\
((\varphi^* C_*^{\ell^1}(H; W))_G)' & = & (\varphi^* C_*^{\ell^1}(H; W)')^G & = & (\varphi^* C_b^*(H; W'))^G \\
\downarrow (C_*^{\ell^1}(\varphi; f)_G)' & & & & \downarrow C_b^*(\varphi; f')^G \\
(C_*^{\ell^1}(G; V)_G)' & \xlongequal{\quad} & (C_*^{\ell^1}(G; V)')^G & \xlongequal{\quad} & C_b^*(G; V')^G
\end{array}$$

$(p \circ C_*^{\ell^1}(\varphi; f)_G)'$ (curved arrow from top-left to bottom-left)
 $(C_*^{\ell^1}(\varphi; f)_G)'$ (vertical arrow from middle-left to bottom-left)

Figure (5.4): Linking ℓ^1 -homology and bounded cohomology of discrete groups (proof of Corollary (5.3))

A straightforward calculation shows that the diagram in Figure (5.4) is a commutative diagram of morphisms of Banach cochain complexes, where all horizontal morphisms are isometric isomorphisms. Thus, applying the translation principle (Theorem (1.1)) to the morphism $p \circ C_*^{\ell^1}(\varphi; f)_G$ of Banach chain complexes proves the first two parts of the corollary. The third part follows because $(V_G)'$ and $(V')^G$ are isometrically isomorphic. \square

An interesting consequence of the third statement is that it provides a characterisation of amenable groups:

Corollary (5.5). *For a discrete group G the following are equivalent:*

- (1) *The group G is amenable.*
- (2) *For all Banach G -modules V , the ℓ^1 -homology $H_*^{\ell^1}(G; V)$ of G with coefficients in V is trivial, i.e., $H_*^{\ell^1}(G; V) \cong H_*^{\ell^1}(1; V_G)$.*

Proof. Amenable groups can be characterised by the vanishing of bounded cohomology with arbitrary (dual) coefficients in non-zero degree [10, 20]. Therefore, the claim follows with help of Corollary (5.3) and Section 3.2.1. \square

Like ℓ^1 -homology of spaces, ℓ^1 -homology of discrete groups cannot see amenable, normal subgroups:

Corollary (5.6). *Let G be a discrete group, let $A \subset G$ be an amenable, normal subgroup, and let V be a Banach G -module. Then the projection $G \rightarrow G/A$ induces an isometric isomorphism*

$$H_*^{\ell^1}(G; V) \cong H_*^{\ell^1}(G/A; V_A).$$

Proof. The corresponding homomorphism

$$H_b^*(G \rightarrow G/A; V'^A \hookrightarrow V'): H_b^*(G/A; V'^A) \rightarrow H_b^*(G; V')$$

is an isometric isomorphism [20, 19; Theorem 1, Corollary 8.5.2] (the case with \mathbf{R} -coefficients was already treated by Ivanov [9; Section 3.8]). Because the inclusion $V'^A \hookrightarrow V'$ is the dual of the projection $V \rightarrow V_A$, we can apply Corollary (5.3). \square

Caveat (5.7). Let G be a discrete group and let $A \subset G$ be an amenable, normal subgroup. Ivanov proved that the cochain complex $C_b^*(G/A)$ is a strong relatively injective G -resolution of the trivial G -module \mathbf{R} [9; Theorem 3.8.4] by showing that the G -morphisms $C_b^*(G/A) \rightarrow C_b^*(G)$ induced by the projection $G \rightarrow G/A$ are split injective [9; Lemma 3.8.1 and Corollary 3.8.2].

Analogously, Park claimed that the G -morphisms $C_n^{\ell^1}(G) \rightarrow C_n^{\ell^1}(G/A)$ are split surjective [21; Lemma 2.4 and Lemma 2.5] and concluded that the $C_n^{\ell^1}(G/A)$ are relatively projective G -modules. Unfortunately, Park's proof [21; proof of Lemma 2.4] contains an error: the A -invariant mean on $B(A, \mathbf{R})$ provided by amenability of A in general is *not* σ -additive.

In fact, $C_n^{\ell^1}(G/A)$ in general is *not* a relatively projective G -module as the following example shows: Let G be an infinite amenable group (e.g., $G = \mathbf{Z}$) and $A := G$. Then the G -action on $G/A = 1$ is trivial. However, since G is infinite, the G -modules $C_n^{\ell^1}(G)$ do not contain any non-zero G -invariant elements. Therefore, any G -morphism of type $C_n^{\ell^1}(G/A) \rightarrow C_n^{\ell^1}(G)$ must be trivial. We now consider the mapping problem

$$\begin{array}{ccc} C_n^{\ell^1}(G/A) = \mathbf{R} & & \\ \swarrow \text{?} & \downarrow \text{id} & \\ C_n^{\ell^1}(G) & \xrightarrow{\pi} & \mathbf{R} \longrightarrow 0 \end{array}$$

with the G -morphism π given by $g_0 \cdot [g_1 | \dots | g_n] \mapsto 1$, which obviously admits a (non-equivariant) split of norm 1; i.e., the morphism π is relatively projective. The argument above shows that this mapping problem cannot have a solution, and hence that $C_n^{\ell^1}(G/A)$ cannot be a relatively projective G -module.

This problem also affects several other results of Park, e.g., her proof of the fact that ℓ^1 -homology depends only on the fundamental group [21; Theorem 4.1] and of the equivalence theorem [21; Theorem 3.7 and 4.4]. \diamond

5.3. ℓ^1 -Homology via projective resolutions. Ivanov proved that bounded cohomology of a topological space with \mathbf{R} -coefficients can be computed in terms of strong relatively injective resolutions [9]. The translation principle allows us to deduce that ℓ^1 -homology of spaces also admits such a description in terms of homological algebra:

Corollary (5.8). *Let X be a countable, connected CW-complex with fundamental group G and let V be a Banach G -module.*

- (1) *There is a canonical isometric isomorphism*

$$H_*^{\ell^1}(X; V) \cong H_*^{\ell^1}(G; V).$$

- (2) *If C is a strong relatively projective resolution of V , then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*

$$H_*^{\ell^1}(X; V) \cong H_*(C_G).$$

- (3) If C is a strong relatively projective resolution of the trivial Banach G -module \mathbf{R} , then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)

$$H_*^{\ell^1}(X; V) \cong H_*((C \otimes V)_G).$$

Therefore, the results of Section 5.2 are also valid for ℓ^1 -homology with twisted coefficients and hence provide generalisations of the results presented in Section 5.1.

Caveat (5.9). Ivanov proved the corresponding theorem for bounded cohomology with \mathbf{R} -coefficients by verifying that $C_b^*(\tilde{X})$ is a strong relatively injective resolution of the trivial Banach G -module \mathbf{R} [9; Theorem 2.4].

The proof that the resolution $C_b^*(\tilde{X})$ is strong relies heavily on the fact that certain chain maps are split injective. However, for the same reasons as explained in Caveat (5.7), it is not possible to translate these arguments into the language of ℓ^1 -chain complexes. Hence, it seems impossible to prove that the chain complex $C_*^{\ell^1}(\tilde{X})$ is a *strong* resolution. In particular, Park’s proof [21; proof of Theorem 4.1] of Corollary (5.8) (with \mathbf{R} -coefficients) is not complete. \diamond

Proof (of Corollary (5.8)). Ad 1. In order to prove the first part of Corollary (5.8), we proceed as follows:

- (1) We establish a connection between $C_*^{\ell^1}(\tilde{X}; V)$ and the strong relatively projective resolution $C_*^{\ell^1}(G; V)$.
- (2) The dual of this morphism, when restricted to the invariants, induces an isometric isomorphism on the level of cohomology of the invariants [13; Appendix B]; this is a straightforward generalisation of Ivanov’s result that bounded cohomology with \mathbf{R} -coefficients can be computed in terms of strong relatively injective resolutions.
- (3) Finally, we apply the translation principle (Theorem (1.1)) to transfer this isometric isomorphism back to ℓ^1 -homology.

First step. Park [21; proof of Theorem 4.1] constructed the following map (“predually” to Ivanov’s construction [9; proof of Theorem 4.1]):

Let $F \subset \tilde{X}$ be a (set-theoretic) fundamental domain of the G -action on \tilde{X} . In the following, the vertices of the standard n -simplex Δ^n are denoted by v_0, \dots, v_n . For a singular simplex $\sigma \in \text{map}(\Delta^n, \tilde{X})$ let $g_0(\sigma), \dots, g_n(\sigma) \in G$ be the group elements defined inductively by the requirement that

$$g_j(\sigma)^{-1} \cdots g_1(\sigma)^{-1} \cdot g_0(\sigma)^{-1} \cdot \sigma(v_j) \in F$$

for all $j \in \{0, \dots, n\}$. Then the map $\eta: C_*^{\ell^1}(\tilde{X}) \rightarrow C_*^{\ell^1}(G)$ given by

$$\begin{aligned} C_n^{\ell^1}(\tilde{X}) &\longrightarrow C_n^{\ell^1}(G) \\ \sigma &\longmapsto g_0(\sigma) \cdot [g_1(\sigma) \mid \dots \mid g_n(\sigma)], \end{aligned}$$

and hence also $\eta_V := \eta \otimes \text{id}_V: C_*^{\ell^1}(\tilde{X}; V) \rightarrow C_*^{\ell^1}(G; V)$, is a morphism of Banach G -chain complexes. Let $(\eta_V)_G: C_*^{\ell^1}(\tilde{X}; V)_G \rightarrow C_*^{\ell^1}(G; V)_G$ denote the morphism of Banach chain complexes induced by η_V .

$$\begin{array}{ccc}
(C_*^{\ell^1}(G; V))' \xrightarrow{(\eta_V)'} (C_*^{\ell^1}(\tilde{X}; V))' & & (C_*^{\ell^1}(G; V)_G)' \xrightarrow{(\eta_V)_G'} (C_*^{\ell^1}(\tilde{X}; V)_G)' \\
\parallel & & \parallel \\
C_b^*(G; V') \xrightarrow{\vartheta_{V'}} C_b^*(\tilde{X}; V') & & (C_*^{\ell^1}(G; V)')^G \xrightarrow{(\eta_V)'^G} (C_*^{\ell^1}(\tilde{X}; V)')^G \\
\parallel & & \parallel \\
C_b^*(G; V')^G \xrightarrow{(\vartheta_{V'})^G} C_b^*(\tilde{X}; V')^G & & C_b^*(G; V')^G \xrightarrow{(\vartheta_{V'})^G} C_b^*(\tilde{X}; V')^G
\end{array}$$

(a) (b)

Figure (5.11): Relating the morphisms η_V and $\vartheta_{V'}$

We verify now that a different choice of fundamental domain $F^* \subset \tilde{X}$ leads to a map chain homotopic to $(\eta_V)_G$: By the fundamental lemma of homological algebra in the context of Banach G -modules [13; Proposition A.7], there is up to G -homotopy only one G -morphism $C_*^{\ell^1}(\tilde{X}) \rightarrow C_*^{\ell^1}(G)$; in fact, $C_*^{\ell^1}(\tilde{X})$ is a Banach G -chain complex consisting of relatively projective G -modules [21; p. 611] and $C_*^{\ell^1}(G)$ is a strong relatively projective resolution of \mathbf{R} [13; Proposition 2.19]. But η and η^* , the map obtained via F^* , are such G -morphisms and hence are G -homotopic. Therefore, also $\eta \otimes \text{id}_V$ and $\eta_V^* := \eta^* \otimes \text{id}_V$ must be G -homotopic, which implies that the induced maps $(\eta_V)_G$ and $(\eta_V^*)_G$ are homotopic. In particular, $H_*((\eta_V)_G): H_*(C_*^{\ell^1}(\tilde{X}; V)_G) \rightarrow H_*(C_*^{\ell^1}(G; V)_G)$ does not depend on the choice of fundamental domain.

Second step. The dual of the G -morphism η_V coincides under the natural isometric isomorphisms $(C_*^{\ell^1}(\tilde{X}; V))' \cong C_b^*(\tilde{X}; V')$ and $(C_*^{\ell^1}(G; V))' \cong C_b^*(G; V')$ of Banach G -cochain complexes with $\vartheta_{V'}: C_b^*(G; V') \rightarrow C_b^*(\tilde{X}; V')$, the morphism of Banach G -cochain complexes given by

$$\begin{aligned}
(5.10) \quad C_b^n(G; V') &\longrightarrow C_b^n(\tilde{X}; V') \\
f &\longmapsto (\sigma \mapsto f(g_0(\sigma), \dots, g_n(\sigma))).
\end{aligned}$$

In other words, the diagram in Figure (5.11)(a) is commutative. Taking G -invariants of this diagram yields the commutative diagram of morphisms of Banach cochain complexes depicted in Figure (5.11)(b).

The restriction $(\vartheta_{V'})^G$ to the subcomplexes of G -invariants induces an isometric isomorphism on the level of cohomology [13; Appendix B]. Hence, also the top row of the diagram (i.e. $(\eta_V)_G'$) must induce an isometric isomorphism on the level of cohomology.

Third step. Therefore, we can derive from the translation principle (Theorem (1.1)) that $(\eta_V)_G: C_*^{\ell^1}(X; V) = C_*^{\ell^1}(\tilde{X}; V)_G \rightarrow C_*^{\ell^1}(G; V)_G$ induces a (canonical) isometric isomorphism on the level of homology. This finishes the proof of the first part.

Ad 2. and 3. These statements follow from the first part combined with the corresponding results on ℓ^1 -homology of discrete groups (Theorem (3.7)). \square

For example, using this description of ℓ^1 -homology via projective resolutions, one can construct a “straightening” on the ℓ^1 -chain complex of countable, connected CW-complexes [13; Section 4.4], generalising the classical straightening of Thurston [28; p. 6.3] in the presence of non-positive curvature. An important aspect of this generalised straightening is that it allows to get control of the semi-norm in measure homology [13; Appendix D], thereby obtaining homological (and hence a bit more transparent) versions of the original proofs [27, 12; Section 4.3, Theorem 1.1 and 1.2] that measure homology and singular homology are *isometrically* isomorphic.

6. SIMPLICIAL VOLUME OF NON-COMPACT MANIFOLDS

The definition of simplicial volume can be adapted to cover also non-compact manifolds. In this section, we demonstrate how to utilise ℓ^1 -homology and the results established in Section 5 to study the simplicial volume of non-compact manifolds: We first express the simplicial volume of non-compact manifolds in terms of ℓ^1 -homology (Section 6.1). In Section 6.2, we present a finiteness criterion for the simplicial volume of non-compact manifolds. Applications of this finiteness criterion are discussed in Section 6.3.

6.1. Simplicial volume – the non-compact case. The ℓ^1 -norm on the singular chain complex admits an obvious extension to the chain complex of locally finite chains (notice however, that there are locally finite chains with *infinite* ℓ^1 -norm). In particular, there is also a notion of simplicial volume for non-compact manifolds:

Definition (6.1) (Simplicial volume of non-compact manifolds). Let M be an oriented, connected (possibly non-compact) n -manifold without boundary. Then the *simplicial volume of M* is defined by

$$\|M\| := \inf \{ \|c\|_1 \mid c \in C_n^{\text{lf}}(M) \text{ locally finite } \mathbf{R}\text{-fundamental cycle of } M \} \in [0, \infty]. \quad \diamond$$

By definition, the simplicial volume of non-compact manifolds is invariant under proper homotopy equivalences. We now provide a description of the simplicial volume for not necessarily compact manifolds in terms of ℓ^1 -homology:

Definition (6.2). If M is an oriented, connected n -manifold without boundary, we write $[M]^{\ell^1} \subset H_n^{\ell^1}(M)$ for the set of all homology classes in $H_n^{\ell^1}(M)$ that are represented by at least one locally finite fundamental cycle (with finite ℓ^1 -norm). \diamond

If M is compact, then the set $[M]^{\ell^1}$ contains exactly one element, namely the class $H_n(C_n(M) \hookrightarrow C_n^{\ell^1}(M))([M])$. However, if M is non-compact, the set $[M]^{\ell^1}$ may be empty (this happens if and only if $\|M\| = \infty$) or consist of more than one element.

Proposition (6.3). Let M be an oriented, connected n -manifold without boundary.

- (1) Then $\|M\| = \inf \{ \|\alpha\|_1 \mid \alpha \in [M]^{\ell^1} \subset H_n^{\ell^1}(M) \}$.
- (2) If $H_n^{\ell^1}(M) = 0$, then $\|M\| \in \{0, \infty\}$.

Proof. The second part is an immediate corollary of the first one. We now prove the first part: Let $j: C_*^{\text{lf}}(M) \cap C_*^{\ell^1}(M) \hookrightarrow C_*^{\ell^1}(M)$ denote the inclusion. By definition,

$$\|M\| = \inf \{ \|\alpha\|_1 \mid \alpha \in H_*(j)^{-1}([M]^{\ell^1}) \}.$$

The sequence $C_*(M) \hookrightarrow C_*^{\text{lf}}(M) \cap C_*^{\ell^1}(M) \hookrightarrow C_*^{\ell^1}(M)$ of inclusions of normed chain complexes shows that the middle complex is a dense subcomplex of the ℓ^1 -chain complex $C_*^{\ell^1}(M)$. Thus, the induced map $H_*(j): H_*(C_*^{\text{lf}}(M) \cap C_*^{\ell^1}(M)) \rightarrow H_*^{\ell^1}(M)$ on homology is isometric (Proposition (2.4)). This yields the desired description of $\|M\|$. \square

For example, if M is an oriented, connected manifold (of non-zero dimension) without boundary and amenable fundamental group, then $\|M\| \in \{0, \infty\}$.

Using the duality principle for semi-norms one also obtains a corresponding result expressing the simplicial volume of non-compact manifolds via bounded cohomology; however, this description is not as convenient as the one in terms of ℓ^1 -homology.

6.2. A finiteness criterion. In general, the simplicial volume of non-compact manifolds is not finite – it can even then be infinite if the manifold in question is the interior of a compact manifold with boundary. In this case, ℓ^1 -homology gives a necessary and sufficient finiteness condition:

Theorem (6.4) (Finiteness criterion). *Let $(W, \partial W)$ be an oriented, compact n -manifold with boundary ∂W . Then the following are equivalent:*

- (1) *The simplicial volume of the interior W° is finite.*
- (2) *The manifold ∂W is ℓ^1 -invisible, i.e.,*

$$H_{n-1}(C_*(\partial W) \hookrightarrow C_*^{\ell^1}(\partial W))([\partial W]) = 0 \in H_{n-1}^{\ell^1}(\partial W).$$

In particular, by combining this finiteness criterion with Proposition (2.4), we obtain Gromov's necessary condition [7; p. 17]: If $\|W^\circ\| < \infty$, then $\|\partial W\| = 0$. Notice that in contrast to Gromov's estimate of the simplicial volume by the minimal volume [7; p. 12, p. 73], the finiteness criterion is purely topological and can be proved by elementary means.

While it is clear that any ℓ^1 -invisible manifold has vanishing simplicial volume by Proposition (2.4), it is an open problem whether any oriented, closed, connected manifold with vanishing simplicial volume is already ℓ^1 -invisible.

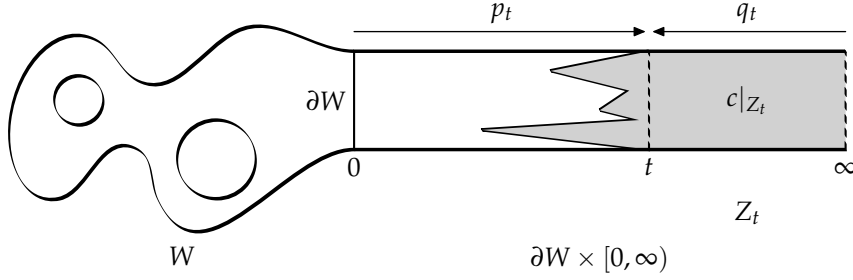
Because the evaluation map linking bounded cohomology and ℓ^1 -homology is continuous, bounded cohomology can detect only whether the semi-norm of a given class in ℓ^1 -homology is zero, but not if the class itself is zero. Therefore, the finiteness criterion as stated above cannot be phrased in terms of bounded cohomology.

Proof (of Theorem (6.4)). The theorem trivially holds if the boundary ∂W is empty; therefore, we assume for the rest of the proof that $\partial W \neq \emptyset$. The homeomorphism [4, 6]

$$W^\circ \cong W \sqcup_{\partial W} \partial W \times [0, \infty) =: M$$

shows that we can look at the notationally more convenient manifold M instead of W° .

1 \Rightarrow 2 Suppose that the simplicial volume $\|W^\circ\| = \|M\|$ is finite. In other words, there is a locally finite fundamental cycle $c = \sum_{j \in \mathbf{N}} a_j \cdot \sigma_j \in C_n^{\text{lf}}(M)$ of M with $\|c\|_1 < \infty$. We now restrict c to a cylinder lying in $\partial W \times [0, \infty) \subset M$. The boundary of this restriction is a fundamental cycle of ∂W and the restriction itself gives rise to the desired boundary in the ℓ^1 -chain complex:

Figure (6.5): The proof of “1 \Rightarrow 2” of the finiteness criterion

More precisely, for $t \in (0, \infty)$ we consider the cylinder $Z_t := \partial W \times [t, \infty)$ and the projections $p_t: \partial W \times [0, \infty) \rightarrow Z_t$ and $q_t: \partial W \times [0, \infty) \rightarrow \partial W \times [0, t]$; the notation is illustrated in Figure (6.5).

Because c is locally finite, there exists a $t \in (0, \infty)$ such that the restriction $c|_{Z_t} \in C_n^{\text{lf}}(M)$ of c to Z_t does not meet W ; by definition, $c|_{Z_t} = \sum_{j \in J_t} a_j \cdot \sigma_j$, where $J_t := \{j \in \mathbf{N} \mid \text{im } \sigma_j \cap Z_t \neq \emptyset\}$. It is not difficult to see that the chain $C_n^{\text{lf}}(p_t)(c|_{Z_t})$ is a relative fundamental cycle of $(Z_t, \partial W \times \{t\})$ and hence that $z_t := \partial(C_n(p_t)(c|_{Z_t}))$ is a fundamental cycle of $\partial W \times \{t\}$. On the other hand, $\|c\|_1$ is finite, so

$$b_t := C_n^{\ell^1}(q_t)(z_t) \in C_n^{\ell^1}(\partial W \times \{t\}).$$

By construction, $\partial b_t = z_t$, which proves that $\partial W \times \{t\}$ is ℓ^1 -invisible. Hence, ∂W is also ℓ^1 -invisible.

2 \Rightarrow 1 Conversely, suppose that part 2 is satisfied, i.e., that ∂W is ℓ^1 -invisible. Therefore, there is a sequence of fundamental cycles $(z_k)_{k \in \mathbf{N}} \subset C_{n-1}(\partial W)$ of ∂W and a sequence of chains $(b_k)_{k \in \mathbf{N}} \subset C_n(W)$ satisfying

$$\begin{aligned} \forall_{k \in \mathbf{N}} \quad \partial b_k &= z_{k+1} - z_k, \\ \sum_{k \in \mathbf{N}} \|b_k\|_1 &< \infty. \end{aligned}$$

By choosing a suitable subsequence of $(z_k)_{k \in \mathbf{N}}$ we can even find two such sequences such that additionally

$$\sum_{k \in \mathbf{N}} \|z_k\|_1 < \infty$$

holds [13; Proposition 6.4]. Now the idea is – similarly to Gromov’s argument in a special case [7; p. 8] – to take a relative fundamental cycle of $(W, \partial W)$ and to glue the $(b_k)_{k \in \mathbf{N}}$ to its boundary. To ensure that the resulting chain is locally finite, we spread out the chain $\sum_{k \in \mathbf{N}} b_k$ over the cylinder $\partial W \times [0, \infty)$.

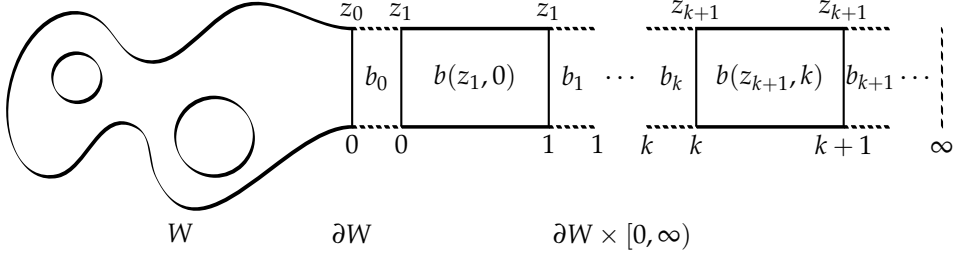


Figure (6.6): The proof of “2 ⇒ 1” of the finiteness criterion

More precisely, let $c \in C_n(W)$ be a relative fundamental cycle of the manifold $(W, \partial W)$ with boundary. Then $\partial c \in C_{n-1}(\partial W)$ is a fundamental cycle of the oriented, compact manifold ∂W . Of course, we may assume that $\partial c = z_0$.

The spreading out of $(b_k)_{k \in \mathbf{N}}$ is achieved by using the following chains: For any cycle $z \in C_{n-1}(\partial W)$ and $k \in \mathbf{N}$ we can find a chain $b(z, k) \in C_n(\partial W \times [0, \infty))$ such that

$$\partial(b(z, k)) = C_{n-1}(j_{k+1})(z) - C_{n-1}(j_k)(z), \quad \|b(z, k)\|_1 \leq n \cdot \|z\|_1;$$

here, $j_k: \partial W \hookrightarrow \partial W \times \{k\} \hookrightarrow \partial W \times [0, \infty)$ denotes the inclusion. For example, such a chain $b(z, k)$ can be constructed by looking at the canonical triangulation of $\Delta^{n-1} \times [0, 1]$ into n -simplices. We set (see also Figure (6.6))

$$b := \sum_{k \in \mathbf{N}} (C_n(j_k)(b_k) + b(z_{k+1}, k))$$

and $\bar{c} := c + b$. Because all b_k and all $b(z_{k+1}, k)$ are finite, the stretched chain b is a well-defined locally finite n -chain of M . Therefore, also $\bar{c} \in C_n^{\text{lf}}(M)$. By construction, \bar{c} is a cycle and $\bar{c}|_{W^\circ} = c|_{W^\circ}$; hence, \bar{c} is a locally finite fundamental cycle of M . Furthermore, $\|\bar{c}\|_1 \leq \|c\|_1 + \|b\|_1$, which shows that $\|M\| < \infty$. \square

6.3. Applications. Before discussing applications of the finiteness criterion (Theorem (6.4)), we first have a tour through the zoo of ℓ^1 -invisible manifolds:

Example (6.7) (ℓ^1 -Invisibility).

- *Vanishing ℓ^1 -homology.* By definition, any oriented, closed n -manifold M satisfying $0 = H_n^{\ell^1}(M) = H_n^{\ell^1}(\pi_1(M))$ is ℓ^1 -invisible. In particular, manifolds with amenable fundamental group are ℓ^1 -invisible.
- *Vanishing bounded cohomology.* Moreover, any oriented, closed n -manifold with $0 = H_b^{n+1}(M) = H_b^{n+1}(\pi_1(M))$ is ℓ^1 -invisible; this follows from the fact that such manifolds satisfy the so-called *uniform boundary condition in degree n* [17, 13; Theorem 2.8, Proposition 6.8]. However, not all ℓ^1 -invisible n -manifolds satisfy the uniform boundary condition in degree n [13; Example 6.9].

$$\begin{array}{ccc}
H_*(M \# N) & \longrightarrow & H_*^{\ell^1}(M \# N) \\
H_m(f) \downarrow & & \cong \downarrow H_m^{\ell^1}(f) \\
H_*(M \vee N) & \longrightarrow & H_*^{\ell^1}(M \vee N) \\
H_*(j_M) \oplus H_*(j_N) \uparrow & & \uparrow H_*^{\ell^1}(j_M) \oplus H_*^{\ell^1}(j_N) \\
H_*(M) \oplus H_m(N) & \longrightarrow & H_*^{\ell^1}(M) \oplus H_*^{\ell^1}(N)
\end{array}
\tag{a}$$

$$\begin{array}{ccc}
H_*(M) & \longrightarrow & H_*^{\ell^1}(M) \\
H_*(p) \downarrow & & \cong \downarrow H_*^{\ell^1}(p) \\
H_*(B) & \longrightarrow & H_*^{\ell^1}(B)
\end{array}
\tag{b}$$

Figure (6.8): Proof of Example (6.7)

- *Functoriality.* Clearly, if $M \rightarrow N$ is a continuous map of non-zero degree between oriented, closed manifolds of the same dimension and if M is ℓ^1 -invisible, then so is N .

Similarly, if the oriented, closed, connected n -manifold M admits a self-map f of non-trivial degree, then M is ℓ^1 -invisible: Let $z \in C_n(M)$ be a fundamental cycle of M and let $b \in C_{n+1}(M)$ with $\partial b = z - 1/\deg f \cdot C_n(f)(z)$. Then

$$\bar{b} := \sum_{k \in \mathbf{N}} \frac{1}{(\deg f)^k} \cdot C_{n+1}(f)^k(b)$$

lies in $C_{n+1}^{\ell^1}(M)$ and $z = \partial \bar{b}$, i.e., M is ℓ^1 -invisible.

- *Products.* If M and N are oriented, closed, connected manifolds, and if M is ℓ^1 -invisible, then using the ℓ^1 -version of the homological cross product on singular chains shows that also the product $M \times N$ is ℓ^1 -invisible.
- *Gluing.* Let M and N be oriented, closed, connected, ℓ^1 -invisible manifolds of the same dimension at least 3. Then the connected sum $M \# N$ is also ℓ^1 -invisible:

Let $j_M: M \rightarrow M \vee N$ and $j_N: N \rightarrow M \vee N$ be the inclusions. The Mayer-Vietoris sequence for $M \vee N$ shows that in non-zero degree $H_*(j_M) \oplus H_*(j_N)$ is an isomorphism mapping $([M], [N])$ to $[M \# N]$. Because M and N are ℓ^1 -invisible, the lowest horizontal map in Figure (6.8)(a) maps $([M], [N])$ to 0.

On the other hand, the pinching map $f: M \# N \rightarrow M \vee N$ induces an isomorphism on the level of fundamental groups and hence induces an isomorphism in ℓ^1 -homology (Corollary (5.2)). Therefore, we can read off the commutative diagram in Figure (6.8)(a) that $M \# N$ is ℓ^1 -invisible.

More generally, the class of ℓ^1 -invisible manifolds of dimension at least 3 is also closed under amenable gluings [13; Proposition 6.10].

- *Fibrations.* If $p: M \rightarrow B$ is a fibration of oriented, closed, connected manifolds whose fibre F is also an oriented, closed, connected manifold of non-zero dimension and if $\pi_1(F)$ is amenable, then M is ℓ^1 -invisible:

A spectral sequence argument yields $\dim B < \dim M - 1$. In particular, $H_*(p)([M]) = 0 \in H_*(B)$. The long exact sequence of homotopy groups associated with the fibration p shows that $\pi_1(p)$ is surjective and that the kernel of $\pi_1(p)$ is a homomorphic image of the amenable group $\pi_1(F)$; thus, $\ker \pi_1(p)$ is amenable [22; Proposition 1.12 and 1.13]. Therefore, $H_*^{\ell^1}(p)$ is an isometric isomorphism (Corollary (5.2)), and we deduce from Figure (6.8)(b) that M is ℓ^1 -invisible.

- *Circle actions.* If M is a smooth, oriented, closed manifold admitting a smooth S^1 -action that is either free or has at least one fixed point, then M is ℓ^1 -invisible:
 - In the first case, we can apply the same argument as for fibrations with amenable fibres because $\pi_1(S^1) \cong \mathbf{Z}$ is amenable.
 - In the second case, it is known that the map on singular homology induced by the classifying map $M \rightarrow B\pi_1(M)$ maps $[M]$ to 0 [7, 16; p. 95, Lemma 1.42], and by Corollary (5.2), the classifying map induces an isometric isomorphism on ℓ^1 -homology.
- *Proportionality.* If M and N are smooth, oriented, closed, connected manifolds equipped Riemannian metrics such that the Riemannian universal coverings of M and N are isometric, then M is ℓ^1 -invisible if and only if N is ℓ^1 -invisible [13; Proposition 6.10]; the proof of this fact is based on an ℓ^1 -version of measure homology.
- *Relation with curvature.* Let M be an oriented, closed, connected Riemannian manifold.
 - If M has positive sectional curvature, then $\pi_1(M)$ is finite [11; Theorem 11.8], hence amenable. In particular, M is ℓ^1 -invisible.
 - If M is flat, then M is ℓ^1 -invisible by proportionality, because any oriented, closed, connected flat manifold has the same Riemannian universal covering as the torus of the same dimension.
 - If M has negative sectional curvature, then $\|M\| \neq 0$ [8] and so M is not ℓ^1 -invisible. \diamond

Equipped with this list of examples of ℓ^1 -invisible manifolds, we apply the finiteness criterion and the description of the simplicial volume of non-compact manifolds in terms of ℓ^1 -homology (Proposition (6.3)) to exhibit a number of simple examples illustrating the simplicial volume of non-compact manifolds:

6.3.1. *Vanishing results.* If $(W, \partial W)$ is an oriented, compact connected n -manifold with ℓ^1 -invisible boundary and $H_n^{\ell^1}(W) = 0$, then $\|W^\circ\| = 0$; this follows from the finiteness criterion (Theorem (6.4)) and Proposition (6.3).

For instance, it follows that $\|\mathbf{R}^n\| = 0$ for all $n \in \mathbf{N}_{>1}$ because the sphere S^{n-1} is ℓ^1 -invisible. On the other hand, the finiteness criterion and $\|S^0\| = 2$ imply that $\|\mathbf{R}\| = \infty$. In particular, the simplicial volume of non-compact manifolds is in general not invariant under homotopy equivalences that are not proper.

Notice that $\|\mathbf{H}^n\| = \|\mathbf{R}^n\| = 0$ for $n \in \mathbf{N}_{>1}$ despite of \mathbf{H}^n being hyperbolic. On the other hand, for certain classes of non-compact, negatively curved manifolds of finite volume non-vanishing results can be proved by more advanced means [14].

6.3.2. *Non-compact manifolds with finite, non-zero simplicial volume.* If M is an oriented, closed, connected manifold with $\|M\| \neq 0$ of dimension $n \geq 2$ (for example, a closed hyperbolic n -manifold), and if N is a non-compact manifold obtained from M by removing a finite number of points, then $0 < \|N\| < \infty$.

This can be seen as follows: By construction, N is the interior of a compact manifold $(N', \partial N')$ whose boundary is a disjoint union of $(n-1)$ -spheres. Because S^{n-1} is ℓ^1 -invisible, the finiteness criterion (Theorem (6.4)) yields $\|N\| < \infty$.

Why is $\|N\|$ non-zero? A straightforward computation shows that $\|N\| \geq \|N', \partial N'\|$ holds [13; Proposition 5.12], where $\|N', \partial N'\|$ is the infimum of the ℓ^1 -norms of all relative fundamental cycles of $(N', \partial N')$. Using the fact that D^n satisfies the uniform boundary condition in degree $n-1$ [17; Theorem 2.8] and that $H_{n-1}(D^n) = 0$, we find a $K \in \mathbf{R}_{>0}$ with the following property: Every relative fundamental cycle $z' \in C_n(N')$ of $(N', \partial N')$ can be extended to a fundamental cycle $z \in C_n(M)$ of M with

$$\|z\|_1 \leq \|z'\|_1 + K \cdot \|\partial z'\|_1 \leq (1 + K \cdot (n+1)) \cdot \|z'\|_1.$$

Therefore, $\|N\| \geq \|N', \partial N'\| \geq 1/(1 + K \cdot (n+1)) \cdot \|M\| > 0$, as claimed.

6.3.3. *Locally symmetric spaces of non-compact type of finite volume.* A class of non-compact manifolds with nice compactification arises naturally in Riemannian geometry, namely in the study of locally symmetric spaces of non-compact type with finite volume.

For instance, if M is an oriented, connected, complete hyperbolic manifold of finite volume, then a classical theorem [1; Corollary D.3.14] asserts that there is an oriented, compact, connected manifold with boundary whose interior is isomorphic to M and whose boundary is a disjoint union of manifolds admitting a flat Riemannian structure. Therefore, $\|M\|$ is finite by the finiteness criterion (Theorem (6.4)).

Now let $M = \Gamma \backslash G/K$ be an oriented, connected, locally symmetric space of non-compact type with finite volume, where Γ is an arithmetic lattice. If the \mathbf{Q} -rank of Γ is equal to 1 or at least 3, then the Borel-Serre compactification of M has ℓ^1 -invisible boundary [13; Section 6.4.5], and hence $\|M\| < \infty$ by the finiteness criterion. However, in view of the more recent, stronger (but also more involved) computations [15, 14] for such manifolds, we do not go into detail here.

6.3.4. *Products.* The simplicial volume of products of two manifolds can be estimated from below by the product of the simplicial volume of the factors if one of the factors is compact [7, 13; p. 17f, Theorem C.7]; however, in the case that the compact factor has vanishing simplicial volume and the other factor has infinite simplicial volume this estimate is inconclusive. In a special case, ℓ^1 -invisibility determines the outcome for such products:

Proposition (6.9). *Let M be an oriented, closed, connected n -manifold. Then*

$$\|M \times \mathbf{R}\| = \begin{cases} 0 & \text{if } M \text{ is } \ell^1\text{-invisible,} \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Because $M \times \mathbf{R}$ is homeomorphic to the interior of the compact manifold $M \times [0, 1]$ with boundary $M \sqcup M$, the finiteness criterion (Theorem (6.4)) shows that $\|M \times \mathbf{R}\|$ is finite if and only if M is ℓ^1 -invisible.

In the case that M is ℓ^1 -invisible, the proof of the finiteness criterion provides us with a locally finite chain $c \in C_{n+1}^{\text{lf}}(M \times [0, \infty))$ such that the sequence $(c_k)_{k \in \mathbf{N}}$ defined by $c_k := c|_{M \times [k, \infty)} \in C_{n+1}^{\text{lf}}(M \times [k, \infty))$ has the following properties:

- For all $k \in \mathbf{N}$ we have $\partial c_k \in C_n(M \times \{k\})$ and c_k is a relative locally finite fundamental cycle of the half-open cylinder $(M \times [k, \infty), M \times \{k\})$.
- Furthermore, $\lim_{k \rightarrow \infty} \|c_k\|_1 = 0$.

For $k \in \mathbf{N}$ we consider the mirror chain $\bar{c}_k := C_{n+1}^{\text{lf}}(\text{id}_M \times r_k)(c_k)$ where $r_k: \mathbf{R} \rightarrow \mathbf{R}$ denotes reflection at k . Then $c_k - \bar{c}_k \in C_{n+1}^{\text{lf}}(M \times \mathbf{R})$ is a locally finite fundamental cycle of $M \times \mathbf{R}$ and $\|M \times \mathbf{R}\| \leq \inf_{k \in \mathbf{N}} \|c_k - \bar{c}_k\|_1 \leq 2 \cdot \inf_{k \in \mathbf{N}} \|c_k\|_1 = 0$. \square

Hence, any oriented, closed, connected manifold with vanishing simplicial volume that is not ℓ^1 -invisible would produce the first example of two manifolds M and N satisfying $\|M\| = 0$, $\|N\| = \infty$ and $\|M \times N\| \neq 0$.

A related problem is to find an example of two non-compact manifolds whose product has non-zero simplicial volume. Using the finiteness criterion we obtain:

Example (6.10). Let $(M, \partial M)$ be an oriented, compact, connected surface of genus at least 1 with non-empty boundary. Then $\|M^\circ \times \mathbf{R}\| = \infty$:

By construction, $M^\circ \times \mathbf{R}$ is the interior of the compact manifold $M \times [0, 1]$ whose boundary is homeomorphic to $M \# M$ and hence is an oriented, closed, connected surface of genus at least 2. Because hyperbolic manifolds are not ℓ^1 -invisible, the finiteness criterion shows that $\|M^\circ \times \mathbf{R}\| = \infty$. \diamond

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