Isomorphisms in ℓ^1-homology

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Abstract. Taking the ℓ^1-completion and the topological dual of the singular chain complex gives rise to ℓ^1-homology and bounded cohomology respectively. Unlike ℓ^1-homology, bounded cohomology is quite well understood by the work of Gromov and Ivanov. We derive a mechanism linking isomorphisms on the level of homology of Banach chain complexes to isomorphisms on the level of cohomology of the dual Banach cochain complexes and vice versa. Therefore, certain results on bounded cohomology can be transferred to ℓ^1-homology. For example, we obtain a new proof that ℓ^1-homology depends only on the fundamental group and that ℓ^1-homology admits a description in terms of projective resolutions.
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1   Introduction

The simplicial volume is a topological invariant of oriented manifolds measuring the complexity of the fundamental class with real coefficients with respect to the ℓ^1-norm: If M is an oriented, closed, connected n-manifold, then the simplicial volume of M is defined as

$$\|M\| := \big\{ \|c\|_1 \;\big|\; c \in C_n(M) \text{ is an } \mathbf{R}\text{-fundamental cycle of } M \big\}.$$

For example, the simplicial volume of spheres and tori is zero, whereas the simplicial volume of negatively curved manifolds is non-zero [4, 14]. Various results indicate that the simplicial volume is a "topological approximation" of the Riemannian volume:

- If M is an oriented, closed, connected, hyperbolic n-manifold, then $\|M\| = \mathrm{vol}(M)/v_n$ [3, 14, 1].
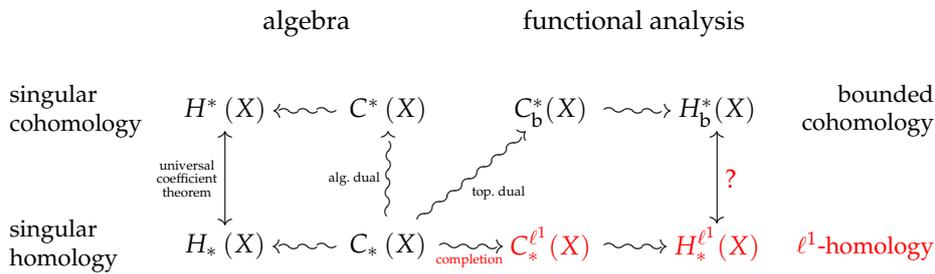
algebra   functional analysis

singular
cohomology
$H^*(X) \longleftarrow\!\!\!\rightsquigarrow C^*(X) \qquad\qquad C_b^*(X) \rightsquigarrow H_b^*(X)$
bounded
cohomology

universal
coefficient   alg. dual   ?
theorem   top. dual

singular
homology
$H_*(X) \longleftarrow\!\!\!\rightsquigarrow C_*(X) \underset{\text{completion}}{\rightsquigarrow} C_*^{\ell^1}(X) \rightsquigarrow H_*^{\ell^1}(X) \qquad \ell^1\text{-homology}$

Figure (1.1): Linking various (co)homology theories related to singular homology

- If M and N are oriented, closed, connected, Riemannian manifolds with isometric Riemannian universal covering, then [3, 14, 13]

$$\frac{\|M\|}{\operatorname{vol}(M)} = \frac{\|N\|}{\operatorname{vol}(N)}.$$

- If M is an oriented, closed, connected, smooth n-manifold, then [3]

$$\|M\| \leq (n-1)^n \cdot n! \cdot \operatorname{minvol}(M).$$

It is almost impossible to calculate the simplicial volume directly. However, Gromov discovered that bounded cohomology can be used to systematically study the simplicial volume [3]. Bounded cohomology is the functional analytic twin of singular cohomology – it is defined using the *topological* dual of the singular chain complex instead of the algebraic one. This change of the underlying cochain complex results in quite peculiar behaviour:

- Bounded cohomology of spaces with amenable fundamental group vanishes (in non-zero degree) [3, 5].

- Bounded cohomology depends only on the fundamental group [3, 5].

- However, there is still a description of bounded cohomology of groups in terms of a certain flavour of homological algebra [5].

The corresponding homology theory is ℓ^1-homology – defined via the ℓ^1-completion of the singular chain complex (cf. Figure (1.1)) – and it is natural to ask whether ℓ^1-homology behaves similarly. For example:

- Does ℓ^1-homology of spaces with amenable fundamental group vanish?

- Does ℓ^1-homology also depend only on the fundamental group?

- Can ℓ^1-homology of groups be described in terms of homological algebra?

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- More generally, how are bounded cohomology and ℓ^1-homology related? Is there some kind of duality?

Matsumoto and Morita [10] found the following duality principle: Bounded cohomology of a space vanishes if and only if its ℓ^1-homology vanishes. In particular, the first question can be answered affirmatively.

Bouarich [2] gave the first proof that ℓ^1-homology depends only on the fundamental group. His proof relies on results in bounded cohomology, the duality principle by Matsumoto and Morita [10] and an ℓ^1-version of Brown's theorem.

We show that applying Matsumoto and Morita's duality principle (generalised to all Banach chain complexes and their duals [6; Proposition 1.2]) to mapping cones of morphisms of Banach chain complexes allows to translate certain results from bounded cohomology to ℓ^1-homology and vice versa (see Theorem (3.1) and Section 4); however, one can show that there is no real duality between ℓ^1-homology and bounded cohomology [7; Remark 3.3].

Our translation mechanism establishes links between ℓ^1-homology and bounded cohomology of discrete groups, as well as between ℓ^1-homology of spaces and ℓ^1-homology of their fundamental group, analogous to Ivanov's result on bounded cohomology. In particular, this gives a new proof that ℓ^1-homology of a space depends only on its fundamental group and since ℓ^1-homology of groups admits a description in terms of homological algebra, the same is true for ℓ^1-homology of spaces. This is explained in Section 4 – more details and more general results can be found in the original article [7].

In comparison to Bouarich's proof, our approach needs more information about bounded cohomology, but seems to be in total more lightweight and yields more general results.

Moreover, Park [9] tried to use an approach similar to Ivanov's work on bounded cohomology to prove, for example, that ℓ^1-homology of spaces depends only on the fundamental group. However, not all of Ivanov's arguments can be carried over to ℓ^1-homology and her proofs contain a significant gap [7; Caveat 5.9 and 6.4].

2   Homology of normed chain complexes

In this section, we introduce the basic objects of study, i.e., normed chain complexes and their homology.

2.1   Normed chain complexes

Definition (2.1).   1. A **normed chain complex** is a chain complex of normed vector spaces, where all boundary morphisms are bounded linear operators. Analogously, a **normed cochain complex** is a cochain complex of normed vector spaces, where all coboundary morphisms are bounded linear operators.

2. A **Banach (co)chain complex** is a normed (co)chain complex consisting of Banach spaces.

3. A **morphism of normed (co)chain complexes** is a (co)chain map between normed (co)chain complexes consisting of bounded operators.   ◇

In this talk, all Banach spaces are Banach spaces over **R** and all (co)chain complexes are indexed over **N**.

Example (2.2). Let X be a topological space.

1. The ℓ^1-**norm** on the singular chain complex $C_*(X)$ with real coefficients is defined as follows: If $c = \sum_{j=0}^{k} a_j \cdot \sigma_j \in C_*(X)$, then

$$\|c\|_1 := \sum_{j=0}^{k} |a_j|.$$

2. The boundary operator $\partial_n \colon C_n(X) \to C_{n-1}(X)$ is – with respect to the ℓ^1-norm – a bounded operator with operator norm $(n+1)$. Hence, $C_*(X)$ is a normed chain complex. Clearly, $C_*(X)$ is in general not complete and thus this complex is no Banach chain complex.

3. Let $p \in (1, \infty]$. Then the singular chain complex with respect to the ℓ^p-norm is *not* a normed chain complex, because the boundary operator is not bounded: For example, for $n \in \mathbf{N}$ let $\sigma_1, \ldots, \sigma_n \colon \Delta^k \longrightarrow X$ be n distinct k-simplices satsifying $\partial\sigma_1 = \cdots = \partial\sigma_n$. We now consider $c_n := 1/n \cdot (\sigma_1 + \cdots + \sigma_n) \in C_k(X)$. By definition, ∂c_n is independent of n, but $\lim_{n\to\infty} \|c_n\|_p = 0$.   ◇

Definition (2.3). Let (C, ∂) be a normed chain complex.

1. The **completion** $(\overline{C}, \overline{\partial})$ of C is the Banach chain complex defined by $(\overline{C})_n := \overline{C_n}$. Since $\partial_n \colon C_n \longrightarrow C_{n-1}$ is a bounded operator, it extends to a bounded operator $\overline{\partial}_n \colon \overline{C_n} \longrightarrow \overline{C_{n-1}}$, which clearly satisfies $\overline{\partial} \circ \overline{\partial} = 0$.

2. The **dual cochain complex** (C', ∂') is the Banach cochain complex defined by

$$\forall_{n\in\mathbf{N}} \quad (C')^n := (C_n)',$$

where $\cdot'$ stands for taking the (topological) dual normed vector space, together with the coboundary operators $(\partial')^n := (\partial_{n+1})'$ and the norm given by $\|f\|_\infty := \sup\{|f(c)| \mid c \in C_n, \|c\| = 1\}$ for $f \in (C')^n$.   ◇

Example (2.4). Let X be a topological space.

1. The ℓ^1-**chain complex of** X is the ℓ^1-completion

$$C_*^{\ell^1}(X) := \overline{C_*(X)}^{\ell^1}$$

of the normed chain complex $C_*(X)$.

2. The Banach cochain complex

$$C_b^*(X) := \left(C_*^{\ell^1}(X)\right)' = \left(C_*(X)\right)'$$

is called the **bounded cochain complex of** X.

3. If $f \colon X \longrightarrow Y$ is a continuous map of topological spaces, then the induced map $C_*(f) \colon C_*(X) \longrightarrow C_*(Y)$ is a chain map that is bounded in each degree (with operator norm equal to 1), i.e., it is a morphism of normed chain complexes. Its extension $C_*^{\ell^1}(f) \colon C_*^{\ell^1}(X) \longrightarrow C_*^{\ell^1}(Y)$ is a morphism of Banach chain complexes and its dual $C_b^*(f) \colon C_b^*(Y) \longrightarrow C_b^*(X)$ is a morphism of Banach cochain complexes. ◇

Other prominent examples of Banach (co)chain complexes are ℓ^1-chain complexes and bounded cochain complexes of discrete groups (see Subsection 4.2).

2.2 Semi-norms on homology

Clearly, the presence of chain complexes calls for the investigation of the corresponding homology. In the case of normed chain complexes, the homology groups carry an additional piece of information – the semi-norm.

Definition (2.5). 1. Let (C, ∂) be a normed chain complex and let $n \in \mathbf{N}$. The **n-th homology of** C is the quotient

$$H_n(C) := \frac{\ker(\partial_n \colon C_n \to C_{n-1})}{\mathrm{im}(\partial_{n+1} \colon C_{n+1} \to C_n)}.$$

2. Dually, if (C, δ) is a normed cochain complex, then its **n-th cohomology** is the quotient

$$H^n(C) := \frac{\ker(\delta^n \colon C^n \to C^{n+1})}{\mathrm{im}(\delta^{n-1} \colon C^{n-1} \to C^n)}.$$

3. Let C be a normed chain complex. Then the norm $\|\cdot\|$ on C induces a semi-norm, also denoted by $\|\cdot\|$, on the homology $H_*(C)$ as follows: If $\alpha \in H_n(C)$, then

$$\|\alpha\| := \inf\{\|c\| \mid c \in C_n, \partial(c) = 0, [c] = \alpha\}.$$

Similarly, we define a semi-norm on the cohomology of normed cochain complexes. ◇

Example (2.6). Let X be a topologica space.

1. The ℓ^1-**homology of** X is defined as

$$H_*^{\ell^1}(X) := H_*\left(C_*^{\ell^1}(X)\right).$$

Dually, the **bounded cohomology of** X is given by

$$H_b^*(X) := H^*\left(C_b^*(X)\right).$$

2. The semi-norms on $H_*^{\ell^1}(X)$ and $H_b^*(X)$ are the ones induced by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively and are also denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively.

3. If $f\colon X \longrightarrow Y$ is a continuous map of pairs of topological spaces, then the maps on ℓ^1-homology and bounded cohomology induced by $C_*^{\ell^1}(f)$ and $C_b^*(f)$ are denoted by $H_*^{\ell^1}(f)$ and $H_b^*(f)$ respectively. $\diamond$

An example of a topological invariant defined in terms of the ℓ^1-semi-norm on singular homology is the simplicial volume:

Example (2.7). Let M be an oriented, closed, connected n-dimensional manifold and let $[M] \in H_n(M)$ be the image of the integral fundamental class of M under the change of coefficients homomorphism. Then the **simplicial volume of M** is defined as

$$\|M\| := \big\|[M]\big\|_1. \qquad\qquad \diamond$$

Because the images of the (co)boundary operators of Banach (co)chain complexes are not necessarily closed, the induced semi-norms on (co)homology need not be norms. Therefore, it is sometimes convenient to look at the corresponding reduced versions $\overline{H}_*(C)$ and $\overline{H}^*(C)$ instead.

Remark (2.8). Any morphism $f\colon C \longrightarrow D$ of normed chain complexes induces linear maps $H_n(f)\colon H_n(C) \longrightarrow H_n(D)$. Since f is continuous in each degree, these maps descend to linear maps $\overline{H}_n(f)\colon \overline{H}_n(C) \longrightarrow \overline{H}_n(D)$. Moreover, the maps $H_n(f)$ and $\overline{H}_n(f)$ are bounded. $\square$

In order to understand semi-norms on the homology of normed chain complexes, it suffices to consider the case of Banach chain complexes. Namely, by approximating boundaries one obtains [7, 11; Proposition 2.7, Lemma 2.9]:

Proposition (2.9). *Let D be a normed chain complex and let C be a dense subcomplex. Then the induced map $H_*(C) \longrightarrow H_*(D)$ is isometric. In particular, the induced map $\overline{H}_*(C) \longrightarrow \overline{H}_*(D)$ must be injective.*

Example (2.10). Let X be a topological space. Then the homomorphism $H_*(X) \longrightarrow H_*^{\ell^1}(X)$ induced by the inclusion $C_*(X) \subset C_*^{\ell^1}(X)$ is isometric with respect to the semi-norms on $H_*(X)$ and $H_*^{\ell^1}(X)$ induced by the ℓ^1-norm.

In particular, if $H_n^{\ell^1}(X) = 0$, then $\|\alpha\|_1 = 0$ for all $\alpha \in H_n(X)$. $\diamond$

Therefore, the simplicial volume can be computed by ℓ^1-homology. Compared to singular homology, ℓ^1-homology and bounded cohomology share the advantage to vanish for a large class of spaces.

3   Duality

The goal is now to find a mechanism that allows to translate results from bounded cohomology to ℓ^1-homology and vice versa. I.e., we are searching for an analog

(or at least an approximation) in the framework of Banach chain complexes of the following classical theorem: *If C is an $\mathbf{R}$-chain complex, then the Kronecker product yields a natural isomorphism*

$$H^*\big(\hom_{\mathbf{R}}(C,\mathbf{R})\big) \cong \hom_{\mathbf{R}}\big(H_*(C),\mathbf{R}\big).$$

Our main result is:

Theorem (3.1). *Let $f\colon C \longrightarrow D$ be a morphism of Banach chain complexes and let $f'\colon D' \longrightarrow C'$ be its dual.*

1. *The induced map $H_*(f)\colon H_*(C) \longrightarrow H_*(D)$ is an isomorphism of vector spaces if and only if $H^*(f')\colon H^*(D') \longrightarrow H^*(C')$ is an isomorphism of vector spaces.*

2. *If $H^*(f')\colon H^*(D') \longrightarrow H^*(C')$ is an isometric isomorphism, then also $H_*(f)$ is an isometric isomorphism.*

First steps towards a proof of this theorem are the following observations by Johnson [6]/Matsumoto and Morita [10], and Gromov [3, 1] respectively:

Theorem (3.2) (Duality principle). *Let C be a Banach chain complex and let C' be its dual. Then $H_*(C)$ vanishes if and only if $H^*(C')$ vanishes.*

This duality principle can be understood as an "absolute" version of duality, whereas Theorem (3.1) can be viewed as a "relative" version. The duality principle shows in particular that ℓ^1-homology is not always trivial.

Theorem (3.3) (Duality principle for semi-norms). *Let C be a normed chain complex and let $n \in \mathbf{N}$. Then*

$$\|\alpha\| = \sup\Big\{ \frac{1}{\|\varphi\|_\infty} \ \Big| \ \varphi \in H^n(C') \text{ and } \langle \varphi, \alpha \rangle = 1 \Big\}$$

holds for each $\alpha \in H_n(C)$. Here, $\sup \varnothing := 0$.

Example (3.4). In particular, the simplicial volume can not only be computed by ℓ^1-homology, but also by bounded cohomology. Historically, this was the first systematic approach to study simplicial volume. ◇

Thus it might be tempting to aim at a result of the type: *Let C be a Banach chain complex. Then the Kronecker product induces a natural isomorphism*

$$H^*(C') \cong \big(H_*(C)\big)' \quad or \quad \overline{H}^*(C') \cong \big(\overline{H}_*(C)\big)'.$$

However, this is not true in general. (It is even wrong in the case "$C = C_*^{\ell^1}(X)$" for certain topological spaces X [12].) Duality statements of the above type can only hold, if the images of the boundary operators are *closed* subspaces – as, for example, in Theorem (3.2).

In order to prove Theorem (3.1), we apply the duality principles (Theorem (3.2) and Theorem (3.3)) to mapping cones.

3.1 Mapping cones

Mapping cones of chain maps are a device translating questions about isomorphisms on homology into questions about the vanishing of homology groups (Lemma (3.6)). Their construction is obviously modelled on the mapping cone (in the category of topological spaces) of continuous maps.

Definition (3.5). 1. Let $f\colon (C,\partial^C) \longrightarrow (D,\partial^D)$ be a morphism of normed chain complexes. Then the **mapping cone** of f, denoted by $\mathrm{Cone}(f)$, is the normed chain complex defined by

$$\mathrm{Cone}(f)_n := C_{n-1} \oplus D_n$$

linked by the boundary operator that is given by the matrix

$$
\begin{pmatrix} -\partial^C & 0 \\ f & \partial^D \end{pmatrix}
\qquad
\begin{array}{c}
\mathrm{Cone}(f)_n \;\; = C_{n-1} \oplus \;\; D_n \\
\downarrow \quad -\partial^C \downarrow \quad f \searrow \quad \downarrow \partial^D \\
\mathrm{Cone}(f)_{n-1} = C_{n-2} \oplus D_{n-1}.
\end{array}
$$

2. Dually, if $f\colon (D,\delta_D) \longrightarrow (C,\delta_C)$ is a morphism of normed cochain complexes, then the **mapping cone** of f, also denoted by $\mathrm{Cone}(f)$, is the normed cochain complex defined by

$$\mathrm{Cone}(f)^n := D^{n+1} \oplus C^n$$

and the coboundary operator determined by the matrix

$$
\begin{pmatrix} -\delta_D & 0 \\ f & \delta_C \end{pmatrix}
\qquad
\begin{array}{c}
\mathrm{Cone}(f)^n \;\; = D^{n+1} \oplus \;\; C^n \\
\downarrow \quad -\delta_D \downarrow \quad f \searrow \quad \downarrow \delta_C \\
\mathrm{Cone}(f)^{n+1} = D^{n+2} \oplus C^{n+1}.
\end{array}
$$

In both cases, we equip the mapping cone with the direct sum of the norms, i.e., the norm given by $\|(x,y)\| := \|x\| + \|y\|$. ◇

Clearly, if f is a morphism of Banach (co)chain complexes, then the mapping cone $\mathrm{Cone}(f)$ is also a Banach (co)chain complex.

The following lemma (of purely algebraic nature) characterises the main feature of mapping cones:

Lemma (3.6). 1. *Let $f\colon C \longrightarrow D$ be a morphism of normed chain complexes. Then the induced map $H_*(f)\colon H_*(C) \longrightarrow H_*(D)$ is an isomorphism (of vector spaces) if and only if all homology groups $H_*(\mathrm{Cone}(f))$ vanish.*

2. *Dually, let $f\colon D \longrightarrow C$ be a morphism of normed cochain complexes. Then the induced map $H^*(f)\colon H^*(D) \longrightarrow H^*(C)$ is an isomorphism if and only if all cohomology groups $H^*(\mathrm{Cone}(f))$ vanish.*

In order to prove Theorem (3.1) it therefore remains to relate the mapping cone of a morphism of normed chain complexes to the mapping cone of its dual.

Lemma (3.7). *Let $f \colon C \longrightarrow D$ be a morphism of normed chain complexes and $f' \colon D' \longrightarrow C'$ the induced morphism between the dual complexes. Then there is a natural isomorphism*

$$\text{Cone}(f)' \cong \Sigma\text{Cone}(f')$$

of normed cochain complexes, relating the mapping cones of f and f'. In particular,

$$H^*\big(\text{Cone}(f)'\big) \cong H^*\big(\Sigma\text{Cone}(f')\big).$$

3.2 Transferring (isometric) isomorphisms

Fusing the properties of mapping cones with the duality principle (Theorem (3.2)) yields a proof of the first part of Theorem (3.1):

Proof (of the first part of Theorem (3.1)). By Lemma (3.6), the map $H_*(f)$ is an isomorphism iff $H_*(\text{Cone}(f)) = 0$. In view of the duality principle (Theorem (3.2)) and Lemma (3.7), this is equivalent to

$$0 = H^*\big(\text{Cone}(f)'\big) = H^*\big(\Sigma\text{Cone}(f')\big) = H^{*-1}\big(\text{Cone}(f')\big).$$

(The duality principle is applicable, because the cone of a morphism of Banach chain complexes is a Banach chain complex.) On the other hand, the cohomlogy groups $H^{*-1}(\text{Cone}(f'))$ are all zero if and only if $f' \colon D' \longrightarrow C'$ is an isomorphism (Lemma (3.6)). $\qquad\square$

Similarly, combining the properties of mapping cones with the duality principle for semi-norms (Theorem (3.3)) proves the second part of Theorem (3.1):

Proof (of the second part of Theorem (3.1)). By the first part, the map $H_*(f)$ is an isomorphism. That this isomorphism is isometric is a consequence of the duality principle for semi-norms (Theorem (3.3)), namely:

Let $n \in \mathbf{N}$ and let $\alpha \in H_n(C)$. Using the duality principle for semi-norms twice and the fact that $H^*(f')$ is an isometric isomorphism, we obtain

$$
\begin{aligned}
\big\|H_n(f)(\alpha)\big\| &= \sup\Big\{ \frac{1}{\|\psi\|_\infty} \;\Big|\; \psi \in H^n(D') \text{ and } \langle \psi, H_n(f)(\alpha)\rangle = 1 \Big\} \\
&= \sup\Big\{ \frac{1}{\|\psi\|_\infty} \;\Big|\; \psi \in H^n(D') \text{ and } \langle H^n(f')(\psi), \alpha\rangle = 1 \Big\} \\
&= \sup\Big\{ \frac{1}{\|H^n(f')(\psi)\|_\infty} \;\Big|\; \psi \in H^n(D') \text{ and } \langle H^n(f')(\psi), \alpha\rangle = 1 \Big\} \\
&= \sup\Big\{ \frac{1}{\|\varphi\|_\infty} \;\Big|\; \varphi \in H^n(C') \text{ and } \langle \varphi, \alpha\rangle = 1 \Big\} \\
&= \|\alpha\|,
\end{aligned}
$$

as desired. $\qquad\square$

However, one cannot expect that a statement of the form "If $H_*(f)$ is an isometric isomorphism, then also $H^*(f')$ is an isometric isomorphism" holds.

4 Applications to ℓ^1-homology

The translation mechanism of Theorem (3.1) allows to transfer certain results from bounded cohomology to ℓ^1-homology. We first investigate ℓ^1-homology of topological spaces, then ℓ^1-homology of discrete groups and finally link these two theories.

4.1 ℓ^1-homology of topological spaces

Using the translation mechanism of Theorem (3.1), we now derive statements concerning isomorphisms in ℓ^1-homology of spaces:

Corollary (4.1). *Let $f\colon X \longrightarrow Y$ be a continuous map of topological spaces.*

1. *The induced homomorphism $H_*^{\ell^1}(f)\colon H_*^{\ell^1}(X) \longrightarrow H_*^{\ell^1}(Y)$ is an isomorphism if and only if $H_b^*(f)\colon H_b^*(Y) \longrightarrow H_b^*(X)$ is an isomorphism.*

2. *If $H_b^*(f)\colon H_b^*(Y) \longrightarrow H_b^*(X)$ is an isometric isomorphism, then $H_*^{\ell^1}(f)$ is also an isometric isomorphism.*

Proof. By definition, $C_b^*(X) = (C_*^{\ell^1}(X))'$ as well as $C_b^*(Y) = (C_*^{\ell^1}(Y))'$ and the cochain map $C_b^*(f)\colon C_b^*(Y) \longrightarrow C_b^*(X)$ coincides with $(C_*^{\ell^1}(f))'$. Applying Theorem (3.1) to $C_*^{\ell^1}(f)$ proves the Corollary. □

Remark (4.2). One can also define ℓ^1-homology and bounded cohomology of pairs of topological spaces. Of course, the above corollary also holds in this setting of relative theories [7]. ◇

Corollary (4.1) allows to transfer certain results from bounded cohomology to ℓ^1-homology. For example, we obtain a new proof of the fact that ℓ^1-homology depends only on the fundamental group and that amenable groups are a blind spot of ℓ^1-homology (Corollary (4.5)):

Definition (4.3). A discrete group A is called **amenable**, if there is a left-invariant mean on the set $B(A, \mathbf{R})$ of bounded functions from A to $\mathbf{R}$, i.e., if there is a linear map $m\colon B(A, \mathbf{R}) \longrightarrow \mathbf{R}$ satisfying

$$\forall_{f \in B(A,\mathbf{R})} \quad \forall_{a \in A} \quad m(f) = m\big(b \mapsto f(a^{-1} \cdot b)\big)$$

and

$$\forall_{f \in B(A,\mathbf{R})} \quad \inf\{f(a) \mid a \in A\} \leq m(f) \leq \sup\{f(a) \mid a \in A\}. \qquad ◇$$

Example (4.4). All finite and all Abelian groups are amenable. Moreover, the class of amenable groups is closed under taking subgroups, quotients, and extensions. An example of a non-amenable group is the free group $\mathbf{Z} * \mathbf{Z}$. ◇

Corollary (4.5) (Mapping theorem for ℓ^1-homology). *The ℓ^1-homology of connected countable CW-complexes depends only on the fundamental group. More generally: Let $f\colon X \longrightarrow Y$ be a continuous map of connected countable CW-complexes such that the induced map $\pi_1(f)\colon \pi_1(X) \longrightarrow \pi_1(Y)$ is surjective and has amenable kernel. Then the induced homomorphism*

$$H_*^{\ell^1}(f)\colon H_*^{\ell^1}(X) \longrightarrow H_*^{\ell^1}(Y)$$

is an isometric isomorphism.

Proof. It is a classical result in the theory of bounded cohomology that in this situation $H_b^*(f)\colon H_b^*(Y) \longrightarrow H_b^*(X)$ is an isometric isomorphism [3, 5; p. 40, Theorem 4.3]. Applying Corollary (4.1) shows that $H_*^{\ell^1}(f)$ is an isometric isomorphism. By looking at the classifying map $X \longrightarrow B\pi_1(X)$ one sees that ℓ^1-homology indeed depends only on the fundamental group (and the classifying map). □

4.2 ℓ^1-homology of discrete groups

Similarly to the results in the previous subsection we can now derive statements concerning isomorphisms in ℓ^1-homology of discrete groups:

Definition (4.6). Let G be a discrete group.

1. The ℓ^1-**chain complex of** G is the ℓ^1-completion $C_*^{\ell^1}(G)$ of the standard bar resolution of G with **R**-coefficients. I.e., if $n \in \mathbf{N}$, then

$$C_n^{\ell^1}(G) := \left\{ \sum_{g \in G^{n+1}} a_g \cdot g_0 \cdot [g_1|\ldots|g_n] \,\middle|\, \forall_{g \in G^{n+1}} \ a_g \in \mathbf{R} \text{ and } \sum_{g \in G^{n+1}} |a_g| < \infty \right\}$$

together with the norm $\left\| \sum_{g \in G^{n+1}} a_g \cdot g_0 \cdot [g_1|\ldots|g_n] \right\|_1 := \sum_{g \in G^{n+1}} |a_g|$ and the G-action characterised by

$$h \cdot \big(g_0 \cdot [g_1|\ldots|g_n] \big) := (h \cdot g_0) \cdot [g_1|\ldots|g_n]$$

for all $g \in G^{n+1}$ and all $h \in G$. The boundary operator is the G-morphism uniquely determined by

$$C_n^{\ell^1}(G) \longrightarrow C_{n-1}^{\ell^1}(G)$$
$$g_0 \cdot [g_1|\ldots|g_n] \longmapsto g_0 \cdot g_1 \cdot [g_2|\ldots|g_n]$$
$$+ \sum_{j=1}^{n-1} (-1)^j \cdot g_0 \cdot [g_1|\ldots|g_{j-1}|g_j \cdot g_{j+1}|g_{j+2}|\ldots|g_n]$$
$$+ (-1)^n \cdot g_0 \cdot [g_1|\ldots|g_{n-1}].$$

2. The **bounded chain complex of** G is the dual

$$C_b^*(G) := C_*^{\ell^1}(G)'.$$

3. The ℓ^1-**homology of** G is the homology of the coinvariants of the ℓ^1-chain complex of G, i.e.,

$$H_*^{\ell^1}(G) := H_*\big(C_*^{\ell^1}(G)_G\big).$$

4. The **bounded cohomology of** G is the cohomology of the invariants of the bounded chain complex of G, i.e.,

$$H_b^*(G) := H^*\big(C_b^*(G)^G\big). \qquad \diamond$$

Notice that the coinvariants V_G of a Banach G-module V (i.e., V is a Banach space with isometric G-action) are defined as

$$V_G := V/\overline{\mathrm{span}\{g \cdot v - v \mid v \in V, g \in G\}}.$$

Remark (4.7). The definition of ℓ^1-homology and bounded cohomology of discrete groups fits also in a more general homological algebraic context. This allows to compute ℓ^1-homology and bounded cohomology of discrete groups via certain projective resolutions and injective resolutions respectively [5, 9]. For example, $C_*^{\ell^1}(G)$ and $C_b^*(G)$ are projective/injective resolutions in this sense [7].

Moreover, one can also introduce ℓ^1-homology and bounded cohomology of discrete groups with coefficients. The above definitions then cover the case with coefficients in the trivial coefficient module $\mathbf{R}$. $\qquad \diamond$

Corollary (4.8). *Let $\varphi \colon H \longrightarrow G$ be a homomorphism of discrete groups.*

1. *Then the homomorphism $H_*^{\ell^1}(\varphi) \colon H_*^{\ell^1}(H) \longrightarrow H_*^{\ell^1}(G)$ is an isomorphism if and only if $H_b^*(\varphi) \colon H_b^*(G) \longrightarrow H_b^*(H)$ is an isomorphism.*

2. *If $H_b^*(\varphi)$ is an isometric isomorphism, then so is $H_*^{\ell^1}(\varphi; f)$.*

Remark (4.9). Of course, this corollary has a generalisation to the case of ℓ^1-homology and bounded cohomology with coefficients. $\qquad \diamond$

Proof. How are $H_*^{\ell^1}(\varphi)$ and $H_b^*(\varphi)$ defined? The homomorphism $\varphi \colon H \longrightarrow G$ induces an H-equivariant morphism $C_*^{\ell^1}(\varphi) \colon C_*^{\ell^1}(H) \longrightarrow \varphi^* C_*^{\ell^1}(G)$ of Banach chain complexes. Here, $\varphi^* C_n^{\ell^1}(G)$ is the Banach space $C_n^{\ell^1}(G)$ together with the H-action induced by φ. Hence, we obtain a morphism

$$C_*^{\ell^1}(\varphi)_H \colon C_*^{\ell^1}(H)_H \longrightarrow \varphi^* C_*^{\ell^1}(G)_H \hookrightarrow C_*^{\ell^1}(G)_G$$

of Banach chain complexes. By definition,

$$H_*^{\ell^1}(\varphi) := H_*\big(C_*^{\ell^1}(\varphi)_H\big) \colon H_*^{\ell^1}(H) \longrightarrow H_*^{\ell^1}(G).$$

On the other hand, $(C_*^{\ell^1}(H)_H)' = (C_*^{\ell^1}(H)')^H = C_b^*(H)^H$ and similar for G. Thus we obtain a morphism $C_*^{\ell^1}(\varphi)'| \colon C_b^*(G) \longrightarrow C_b^*(H)$ and by definition

$$H_b^*(\varphi) := H^*\big(C_*^{\ell^1}(\varphi)'|\big) \colon H_b^*(G) \longrightarrow H_b^*(H).$$

Now the commutative diagram

$$
\begin{array}{ccc}
\left(\varphi^* C_*^{\ell^1}(G)_H\right)' & \xrightarrow{\ (C_*^{\ell^1}(\varphi)_H)'\ } & \left(C_*^{\ell^1}(H)_H\right)' \\
\| & & \| \\
\varphi^* C_{\mathsf{b}}^*(G)^H = \left(\varphi^* C_*^{\ell^1}(G)'\right)^H & \xrightarrow[\ C_*^{\ell^1}(\varphi)'|\ ]{} & \left(C_*^{\ell^1}(H)'\right)^H = C_{\mathsf{b}}^*(G)^H.
\end{array}
$$

together with Theorem (3.1) applied to $C_*^{\ell^1}(\varphi)_H$ proves the corollary. □

Corollary (4.8) enables us to carry over many results on bounded cohomology to ℓ^1-homology. A small example of this procedure is the following:

Corollary (4.10). *Let G be a discrete group, let $A \subset G$ be an amenable normal subgroup. Then the projection $G \longrightarrow G/A$ induces an (isometric) isomorphism*

$$H_*^{\ell^1}(G) \cong H_*^{\ell^1}(G/A).$$

Proof. The corresponding homomorphism

$$H_{\mathsf{b}}^*(G \twoheadrightarrow G/A)\colon H_{\mathsf{b}}^*(G/A) \longrightarrow H_{\mathsf{b}}^*(G)$$

is an isometric isomorphism [5; Section 3.8]. Now Corollary (4.8) finishes the proof.
□

4.3 ℓ^1-homology via projective resolutions

Ivanov developed a homological algebraic approach to bounded cohomology [5] and showed that bounded cohomology of spaces coincides with bounded cohomology of groups and hence can be computed by certain injective resolutions. Similarly, by applying the translation mechanism of Theorem (3.1) to an appropriate chain map

$$C_*^{\ell^1}(X) \longrightarrow C_*^{\ell^1}\left(\pi_1(X)\right)_{\pi_1(X)},$$

we can deduce that ℓ^1-homology of a space coincides with the ℓ^1-homology of the fundamental group. Hence, ℓ^1-homology of spaces admits also a description in terms of projective resolutions:

Corollary (4.11). *Let X be a countable connected CW-complex with fundamental group G.*

1. *There is a canonical isometric isomorphism*

$$H_*^{\ell^1}(X) \cong H_*^{\ell^1}(G).$$

2. *In particular: If C is a strong relatively projective resolution of the trivial Banach G-module $\mathbf{R}$, then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*

$$H_*^{\ell^1}(X) \cong H_*(C_G).$$

Remark (4.12). Similarly to singular homology, there is also a version of ℓ^1-homology with twisted coefficients and there is also a corresponding version of Corollary (4.11) with twisted coefficients [7; Theorem 6.3]. ◇

Proof (of Corollary (4.11)). 1. In order to prove the first part of the corollary, we proceed as follows:

a) For comparing $H_*^{\ell^1}(X)$ and $H_*(C_*^{\ell^1}(G)_G)$, it is convenient to write also $H_*^{\ell^1}(X)$ in the form $H_*(C_G)$ for some Banach G-chain complex C.

   As Park observed [9, 7; proof of Theorem 4.1, Proposition 6.2], the universal covering map $\widetilde{X} \longrightarrow X$ induces an isometric isomorphism

$$C_*^{\ell^1}(\widetilde{X})_G \longrightarrow C_*^{\ell^1}(X)$$

   of Banach chain complexes. In particular, we obtain a canonical isometric isomorphism

$$H_*^{\ell^1}(X) \cong H_*\big(C_*^{\ell^1}(\widetilde{X})_G\big).$$

b) Using a fundamental domain of the G-action on $\widetilde{X}$ one can explicitly construct a G-equivariant morphism

$$\eta \colon C_*^{\ell^1}(\widetilde{X}) \longrightarrow C_*^{\ell^1}(G)$$

   of Banach chain complexes. Clearly, it suffices to show that the corresponding morphism $\eta_G \colon C_*^{\ell^1}(\widetilde{X})_G \longrightarrow C_*^{\ell^1}(G)_G$ induces an isometric isomorphism $H_*(C_*^{\ell^1}(\widetilde{X})_G) \longrightarrow H_*(C_*^{\ell^1}(G)_G)$.

c) The idea is to apply Theorem (3.1) to the morphism η_G. Therefore, it is necessary to investigate the properties of the dual $(\eta_G)'$: Ivanov showed that the lower horizontal arrow in the commutative diagram

$$
\begin{array}{ccc}
\big(C_*^{\ell^1}(G)_G\big)' & \xrightarrow{(\eta_G)'} & \big(C_*^{\ell^1}(\widetilde{X})_G\big)' \\
\| & & \| \\
\big(C_*^{\ell^1}(G)'\big)^G & \xrightarrow{(\eta')^G} & \big(C_*^{\ell^1}(\widetilde{X})'\big)^G \\
\| & & \| \\
C_b^*(G)^G & \longrightarrow & C_b^*(\widetilde{X})^G
\end{array}
$$

   induces an isometric isomorphism on the level of cohomology [5; proof of Theorem 4.1].

d) Finally, we apply Theorem (3.1) to the morphism η_G, which allows us to deduce that η_G induces an isometric isomorphism on the level of homology.

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2. Because $C_*^{\ell^1}(G)$ is a strong relatively projective resolution of $\mathbf{R}$, standard methods from homological algebra [7; Appendix A] provide us with a canonical isomorphism

$$H_*(C_G) \cong H_*\big(C_*^{\ell^1}(G)_G\big).$$

Thus the first part yields $H_*^{\ell^1}(X) \cong H_*(C_G)$, as was to be shown.

(This isomorphism is in general *not* isometric – the definition of strong relatively projective resolutions allows to scale the norm.) $\square$

5 Making a case for ℓ^1-homology

The results presented so far might indicate that ℓ^1-homology is merely a shadow of bounded cohomology. But in the setting of non-compact manifolds there are also genuine applications of ℓ^1-homology [8]:

The **simplicial volume** of an oriented, connected, not necessarily compact manifold V is defined by the ℓ^1-semi-norm of the fundamental class of V in *locally finite* singular homology [3].

In particular, the simplicial volume of non-compact manifolds can be infinite. For a special type of non-compact manifolds there is a finiteness criterion in terms of ℓ^1-homology:

Theorem (5.1). *Let $(W, \partial W)$ be an oriented, compact, connected n-manifold with boundary and let $V := W^\circ$. Then the following are equivalent:*

1. *The simplicial volume $\|V\|$ is finite.*

2. *The fundamental class of ∂W vanishes in ℓ^1-homology, i.e.,*

$$H_{n-1}(i_{\partial W})\big([W]\big) = 0 \in H_{n-1}^{\ell^1}(\partial W).$$

Here, $i_{\partial W}$ denotes the canonical inclusion $C_*(\partial W) \hookrightarrow C_*^{\ell^1}(\partial W)$.

This phenomenon is probably not visible on the level of bounded cohomology, because the Kronecker product linking bounded cohomology and ℓ^1-homology is not able to distinguish between ℓ^1-homology and reduced ℓ^1-homology and therefore cannot detect the vanishing of a certain class in unreduced ℓ^1-homology.

This finiteness criterion is a generalisation of Gromov's necessary condition "If $\|V\| < \infty$, then $\|\partial W\| = 0$" [3].

Example (5.2). Since $\|S^0\| = 2$, we obtain $\|\mathbf{R}\| = \infty$. On the other hand, if $n \in \mathbf{N}_{>1}$, then $\|\mathbf{R}^n\| < \infty$. More precisely, if $n \in \mathbf{N}_{>1}$, then $\|\mathbf{R}^n\| = 0$ [3]. $\diamond$

Example (5.3). If V is a complete, connected, hyperbolic manifold of finite volume, then $\|V\|$ is finite. $\diamond$

Moreover, in case the simplicial volume of a non-compact manifold is finite, it is easier to see how to calculate it in terms of ℓ^1-homology than in terms of bounded cohomology.

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