Simplicial Volume

Clara Strohm[1] WS 2003/04 Diplomandenseminar Topologie

The simplicial volume of an oriented closed connected manifold is a measure for the efficiency of representing its fundamental homology class by singular simplices (with real coefficients). Its first appearance is in Gromov's proof of the Mostow Rigidity Theorem. After defining the simplicial volume and deducing some of its basic properties we will introduce bounded cohomology as a tool for examining the simplicial volume. Moreover, we will explore some of the relations of the simplicial volume to other volumes in differential geometry. In the last section we will discuss the following conjecture of Gromov ([Gromov91, Section 8A]):

Conjecture. *Let M be an aspherical closed orientable manifold of dimension at least 1, whose simplicial volume vanishes. Then*

$$\forall_{p\in\mathbf{N}} \quad b_p^{(2)}\big(\widetilde{M}\big) = 0.$$

[1] strohm@math.uni-muenster.de
http://omega.sdf-eu.org/sv_sem/

Contents

Basic notation

For simplicity we always assume that manifolds have dimension at least 1. The symbols $\mathbf{Z}, \mathbf{N}, \mathbf{R}$ denote the set of integers, the set of nonnegative integers and the set of real numbers, respectively. The one point space is written as $\bullet$.

1 ■ Simplicial Volume

In this section the simplicial volume of an oriented closed connected manifold will be defined, and some basic properties will be deduced.

The most other results about the simplicial volume are derived from the theory of bounded cohomology or from differential geometry. We will discuss these approaches in later sections.

Since the simplicial volume and bounded cohomology are living in the world of singular (co)homology, basic definitions and notations from the singular theory are recapitulated in 1.1.

1.1 ■ Singular (co)homology refreshed

The goal of this paragraph is to recall the basic definitions and notations of singular (co)homology.

The key objects of singular (co)homology are simplices:

Definition (1.1). For $k \in \mathbf{N}$ the set

$$\Delta^k := \left\{ (x_0, \dots, x_k) \in \mathbf{R}^{k+1}_{\geq 0} \,\middle|\, \sum_{j=0}^{k} x_j = 1 \right\}$$

is called the **k-dimensional standard simplex**. For $j \in \{0, \dots, k\}$ we let

$$\begin{aligned} \partial_j \colon \Delta^{k-1} &\longrightarrow \Delta^k \\ (x_0, \dots, x_{k-1}) &\longmapsto (x_0, \dots, x_{j-1}, 0, x_j, \dots, x_{k-1}) \end{aligned}$$

be the inclusion of the **j-th face of Δ^k**. $\diamond$

To be precise, the face mappings ∂_j also depend on k, but we omit k in the notation since it can easily be inferred from the context.

The main idea of the singular theory is to examine continuous maps from the standard simplices into a topological space to detect the "holes" of this space.

Definition (1.2). Let X be a topological space and $k \in \mathbf{N}$.

- The set $S_k(X)$ of **singular k-simplices** is the set of all continuous functions $\Delta^k \longrightarrow X$. If $\sigma \in S_k(X)$ and $j \in \{0, \dots, k\}$, then $\sigma \circ \partial_j$ is called the **j-th face of σ**.

- Let R be a commutative ring with unit. Then the set $C_k(X, R)$ of **singular k-chains of X with coefficients in R** is the free R-module generated by $S_k(X)$.

- The **boundary operator** $\partial\colon C_{k+1}(X, R) \longrightarrow C_k(X, R)$ is the R-module homomorphism given on the generators $\sigma \in S_{k+1}(X)$ by

$$\partial(\sigma) := \sum_{j=0}^{k+1} (-1)^j \cdot \sigma \circ \partial_j.$$

  The elements of $\operatorname{im} \partial$ are called – as geometry suggests – **boundaries**, and the elements of $\ker \partial$ are called **cycles**.

- We define the set $C^k(X, R)$ of **singular k-cochains of X with coefficients in R** as the algebraic dual of $C_k(X, R)$, i.e.

$$C^k(X, R) := \operatorname{Hom}_{\mathbf{Z}}\big(C_k(X), R\big) = \operatorname{Hom}_R\big(C^k(X, R), R\big)$$

- The boundary operator $\partial\colon C_{k+1}(X, R) \longrightarrow C_k(X, R)$ induces the **coboundary operator** $\delta\colon C^k(X, R) \longrightarrow C^{k+1}(X, R)$ as follows

$$\begin{aligned} \delta\colon C^k(X, R) &\longrightarrow C^{k+1}(X, R) \\ f &\longmapsto \big(c \mapsto (-1)^{k+1} \cdot f(\partial c)\big). \end{aligned}$$

  The elements of $\operatorname{im} \delta$ are called **coboundaries** and the elements of $\ker \delta$ are called **cocycles**.

In the special case $R = \mathbf{Z}$ we use the abbreviations $C_k(X) := C_k(X, \mathbf{Z})$, et cetera. ◇

A somewhat nasty calculation shows that in the situation of the above definition the equality $\partial \circ \partial = 0$ holds (i.e. the singular chains form a chain complex). Therefore $\delta \circ \delta = 0$, too. Hence the following definition makes sense.

Definition (1.3). Let X be a topological space, $k \in \mathbf{N}$, and R be a commutative ring with unit.

- The R-module

  $$H_k(X,R) := {}^{\ker\left(\partial\colon C_k(X,R)\to C_{k-1}(X,R)\right)}/_{\operatorname{im}\left(\partial\colon C_{k+1}(X,R)\to C_k(X,R)\right)}$$

  is called **k-th singular homology of X with coefficients in R.**

- The R-module

  $$H^k(X,R) := {}^{\ker(\delta\colon C^k(X,R)\to C_{k+1}(X,R))}/_{\operatorname{im}(\delta\colon C^{k-1}(X,R)\to)C^k(X,R)}$$

  is called **k-th singular cohomology of X with coefficients in R.**

For brevity we write

$$H_k(X) := H_k(X,\mathbf{Z}),$$
$$H^k(X) := H^k(X,\mathbf{Z}).$$ ◇

1.2 ■ Definition of the simplicial volume

We will first introduce a norm on the singular chain groups with real coefficients, which induces a seminorm on singular homology. The simplicial volume of an oriented closed connected manifold is then defined as the seminorm of the fundamental homology class.

For convenience we introduce the following notation:

Definition (1.4). If M is an R-module, $B \subset M$ and $(a_b)_{b\in B} \subset R$, we write $\sum_{b\in B}^{\text{fin}} a_b \cdot b \in M$ to make clear that all but a finite number of the coefficients $(a_b)_{b\in B}$ are 0. ◇

Definition (1.5). Let X be a topological space, $k \in \mathbf{N}$ and $a = \sum_{\sigma\in S_k(X)}^{\text{fin}} a_\sigma \cdot \sigma \in C_k(X,\mathbf{R})$. Then

$$\|a\|_1 := \sum_{\sigma\in S_k(X)}^{\text{fin}} |a_\sigma|.$$ ◇

Since the sum is finite this norm is well-defined an turns the singular chain group $C_k(X, \mathbf{R})$ into a normed real vector space.

Definition (1.6). Let X be a topological space, $k \in \mathbf{N}$ and $\alpha \in H_k(X, \mathbf{R})$ Then $\|\cdot\|_1$ induces a semi-norm as follows:

$$\|\alpha\|_1 := \inf\{\|a\|_1 \mid a \in C_k(X, \mathbf{R}), \partial(a) = 0, [a] = \alpha\}. \qquad \diamond$$

This seminorm measures how efficient a homology class can be represented by singular chains.

Definition (1.7). Let M be an oriented closed connected n-dimensional manifold. We will denote the corresponding fundamental homology class of M by $[M] \in H_n(M)$. Moreover, we write $[M]_{\mathbf{R}}$ for the image of $[M]$ under the change-of-coefficients homomorphism $H_n(M, \mathbf{Z}) \longmapsto H_n(M, \mathbf{R})$. $\diamond$

The fundamental homology class $[M]$ of an n-dimensional oriented closed connected manifold M is the generator of $H_n(M) \cong \mathbf{Z}$ which is compatible with the orientation on M.

A good geometric description of $[M]$ can be found in [Bredon, p. 338f.]: If $\sigma_1, \ldots, \sigma_m \colon \Delta^n \longrightarrow M$ is a triangulation of M by singular simplices, then $\sum_{j=1}^m \sigma_j$ represents $[M]$.

Definition (1.8). The **simplicial volume** of an oriented closed connected manifold M is defined as

$$\|M\| := \big\|[M]_{\mathbf{R}}\big\|_1. \qquad \diamond$$

In Section 4.5 we will investigate the relation of the simplicial volume to other volumes. This will indicate why the simplicial volume indeed is something like a volume.

Lemma (1.9). *The simplicial volume of the circle S^1 equals 0.*

Proof. There are more sophisticated techniques for computing $\|S^1\|$, but to emphasise the geometric intuition we will sketch the calculation "by hand":

For $k \in \mathbf{N}$ we define

$$\begin{aligned} \sigma_k \colon \Delta^1 &\longrightarrow S^1 \\ (t, t-1) &\longmapsto e^{k \cdot 2i\pi t}, \end{aligned}$$

i.e. σ_k wraps the standard 1-simplex k times around S^1. Then σ_k and $k \cdot \sigma_1$ are homologous cycles in $C_1\left(S^1, \mathbf{R}\right)$ and $\left[S^1\right]_{\mathbf{R}} = \sigma_1$ by the Hurewicz Theorem. Therefore

$$\begin{aligned} \left\|S^1\right\| &\leq \inf_{k \in \mathbf{N}} \left\| \frac{1}{k} \cdot \sigma_k \right\|_1 = \inf_{k \in \mathbf{N}} \frac{1}{k} \\ &= 0. \end{aligned}$$ □

It is also possible to define the simplicial volume for non-compact oriented manifolds (via locally finite homology – cf. [Lück, Chapter 14], [Gromov, p. 8]).

1.3 ■ Elementary properties of the simplicial volume

In the following section some elementary properties (such as homotopy invariance and multiplicativity) of the simplicial volume are deduced from the definition. Moreover, some examples for manifolds with vanishing simplicial volume are given. It is more difficult to show that a manifold has positive simplicial volume. Such examples are studied in Section 4.

Lemma (1.10). *Let $f\colon X \longrightarrow Y$ be a continuous map of topological spaces, $k \in \mathbf{N}$ and $\alpha \in H_k(X, \mathbf{R})$. Then we get for the induced seminorm*

$$\left\|H_k\left(f, \mathbf{R}\right)(\alpha)\right\|_1 \leq \|\alpha\|_1 .$$

Proof. Let $\sum^{\text{fin}}_{\sigma \in S_k(X)} a_\sigma \cdot \sigma \in C_k\left(X, \mathbf{R}\right)$ be a cycle representing α. Then $H_k\left(f, \mathbf{R}\right)(\alpha)$ is represented by $\sum^{\text{fin}}_{\sigma \in S_k(X)} a_\sigma \cdot f \circ \sigma$. Therefore

$$\begin{aligned} \left\|H_k\left(f, \mathbf{R}\right)(\alpha)\right\|_1 &= \left\| \left[\sum^{\text{fin}}_{\sigma \in S_k(X)} a_\sigma \cdot f \circ \sigma \right] \right\|_1 \leq \left\| \sum^{\text{fin}}_{\sigma \in S_k(X)} a_\sigma \cdot f \circ \sigma \right\|_1 \\ &\leq \sum^{\text{fin}}_{\sigma \in S_k(X)} |a_\sigma| = \left\| \sum^{\text{fin}}_{\sigma \in S_k(X)} a_\sigma \cdot \sigma \right\|_1 . \end{aligned}$$

Taking the infimum over all cycles representing α yields the desired inequality $\left\|H_k(f)(\alpha)\right\|_1 \leq \|\alpha\|_1$. □

One of the main characteristics of the simplicial volume is that it is multiplicative with respect to the degree of maps.

Definition (1.11). Let M and N be oriented closed connected n-dimensional manifolds and let $[M] \in H_n(M), [N] \in H_n(N)$ be their fundamental homology classes. Then the **degree** of a continuous map $f\colon M \longrightarrow N$ is the integer $\deg(f) \in \mathbf{Z}$ with
$$H_n(f)\big([M]\big) = \deg(f) \cdot [N]. \qquad \diamond$$

Remark (1.12). In the situation of the definition,
$$H_n(f, \mathbf{R})\big([M]_{\mathbf{R}}\big) = \deg(f) \cdot [N]_{\mathbf{R}}$$
also holds (by the Universal Coefficient Theorem).

Lemma (1.13). *Let $f\colon M \longrightarrow N$ be a map of orientable closed connected manifolds of the same dimension. Then*
$$\|M\| \geq |\deg f| \cdot \|N\|.$$

Proof. By the previous lemma and the remark we have
$$\begin{aligned} \|M\| &= \big\|[M]_{\mathbf{R}}\big\|_1 \\ &\geq \big\|H_*(f, \mathbf{R})([M]_{\mathbf{R}})\big\|_1 \\ &= \big\|\deg(f) \cdot [N]_{\mathbf{R}}\big\|_1 \\ &= |\deg(f)| \cdot \|N\|. \end{aligned}$$
□

Corollary (1.14). *Because homotopy equivalences induce maps of degree 1 or -1, it follows that the simplicial volume is a homotopy invariant.* □

Corollary (1.15). *If $f\colon M \longrightarrow M$ is a self-map of an orientable closed connected manifold with $|\deg f| \geq 2$, then*
$$\|M\| = 0.$$
In particular $\|S^n\| = 0$ for all $n \in \mathbf{N}_{\geq 1}$. □

Example. The condition "aspherical" in Gromov's conjecture is necessary: The manifold S^2 is closed and orientable, but not aspherical since $\pi_2(S^2) \cong \mathbf{Z}$. By Corollary (1.15) we have $\|S^2\| = 0$. On the other hand [Lück, Theorem 6.54(8)]
$$b_0^{(2)}(S^2) = \frac{1}{\pi_1(S^2)} = 1 \neq 0.$$

Corollary (1.16). *Let $p\colon M \longrightarrow N$ be a d-sheeted smooth covering of orientable closed connected manifolds. Then*
$$\|M\| = d \cdot \|N\|.$$

Proof. Let $n := \dim N$. Since p is a local homeomorphism, it follows also that $\dim M = n$. By [Dold, Proposition VIII.4.7] we know that

$$\deg p = d.$$

Hence, $\|M\| \geq d \cdot \|N\|$. Why does "$\leq$" hold as well? For $\sigma \in S_n(N)$ let $L(\sigma)$ be the set of singular simplices $\widetilde{\sigma} \in S_n(M)$ with $p \circ \widetilde{\sigma} = \sigma$. Since p is a d-sheeted covering and Δ^n is simply connected, the set $L(\sigma)$ contains precisely d elements. If $a = \sum_{\sigma \in S_n(N)}^{\text{fin}} a_\sigma \cdot \sigma \in C_n(N, \mathbf{R})$ represents $[N]_{\mathbf{R}}$, then

$$\widetilde{a} := \sum_{\sigma \in S_n(N)}^{\text{fin}} a_\sigma \cdot \sum_{\widetilde{\sigma} \in L(\sigma)} \widetilde{\sigma}$$

represents $[M]_{\mathbf{R}}$, because $\partial(\widetilde{a}) = 0$ and

$$\begin{aligned} H_n(p, \mathbf{R})\big([\widetilde{a}]\big) &= \Bigg[\sum_{\sigma \in S_n(N)}^{\text{fin}} a_\sigma \cdot d \cdot \sigma \Bigg] \\ &= d \cdot [N]_{\mathbf{R}} \\ &= H_n(p, \mathbf{R})\big([M]_{\mathbf{R}}\big) \end{aligned}$$

and $d \geq 1$. Thus

$$\begin{aligned} \|M\| = \big\|[M]_{\mathbf{R}}\big\|_1 \leq \|\widetilde{a}\|_1 \leq d \cdot \sum_{\sigma \in S_n(N)}^{\text{fin}} |a_\sigma| \\ = d \cdot \|a\|_1, \end{aligned}$$

which implies $\|M\| \leq d \cdot \big\|[N]_{\mathbf{R}}\big\|_1 = d \cdot \|N\|$. □

Until now we have not encountered a manifold whose simplicial volume does *not* vanish. Some examples for such manifolds are given in Section 4.5.

The main tools for studying for further properties of the simplicial volume are:

- *Bounded Cohomology.* Bounded cohomology is the functional analytic twin of singular cohomology – it is constructed similar to singular cohomology from singular homology using the topological dual space instead of the algebraic one. Since the simplicial volume is related to some sort of ℓ^1-norm (which is dual to the ℓ^∞-norm), it seems plausible that there is a connection between bounded cohomology and the simplicial volume.

- *Differential geometry.* Although the simplicial volume is defined topologically it is related to sectional curvature and other volumes appearing in differential geometry.

We will explore these approaches in the following sections.

2 ■ Bounded Cohomology

Bounded cohomology is the functional analytic twin of singular cohomology – it is constructed similar to singular cohomology from singular homology using the topological dual space instead of the algebraic one. Furthermore, there is also a notion of bounded group cohomology.

After defining bounded cohomology we will have a look at its astonishing properties:

- The bounded cohomology of a topological space depends only on its fundamental group.
- Bounded cohomology cannot detect amenable groups.
- Bounded cohomology does not satisfy the excision axiom.
- The first bounded cohomology group always vanishes.

2.1 ■ Definition of Bounded Cohomology

To define bounded cohomology we need a norm on singular cochains.

Definition (2.1). Let X be a topological space and $k \in \mathbf{N}$. For a singular cochain $f \in C^k(X, \mathbf{R})$ the **supremum norm** is defined via

$$\|f\|_\infty := \sup_{\sigma \in S_k(X)} |f(\sigma)|.$$

This induces the following **seminorm on** $H^k(X, \mathbf{R})$ by

$$\forall_{\varphi \in H^k(X,\mathbf{R})} \quad \|\varphi\|_\infty := \inf\{\|f\|_\infty \mid f \in C^k(X, \mathbf{R}), \delta(f) = 0, [f] = \varphi\}. \qquad \diamond$$

Definition (2.2). Let X be a topological space and $k \in \mathbf{N}$. Then the vector space of **bounded k-cochains** is given by

$$\widehat{C}^k(X) := \{f \in C^k(X, \mathbf{R}) \mid \|f\|_\infty < \infty\}. \qquad \diamond$$

It is easy to see that the coboundary operator on $C^k(X, \mathbf{R})$ restricts to bounded cochains, i.e.

$$\delta\big(\widehat{C}^k(X)\big) \subset \widehat{C}^{k+1}(X).$$

Thus $\widehat{C}^*(X)$ again is a cochain complex.

Definition (2.3). Let X be a topological space and $k \in \mathbf{N}$. The **k-th bounded cohomology group of** X is defined by

$$\widehat{H}^k(X) := \ker\big(\delta|_{\widehat{C}^k(X)} \colon \widehat{C}^k(X) \longrightarrow \widehat{C}^{k+1}(X)\big) / \operatorname{im}\big(\delta|_{\widehat{C}^{k-1}(X)} \colon \widehat{C}^{k-1}(X) \longrightarrow \widehat{C}^k(X)\big). \qquad \diamond$$

As with the simplicial volume it is not immediately clear from the definition that there are spaces whose bounded cohomology does not vanish. An example is given in Section 2.6.

Definition (2.4). Let X be a topological space and $k \in \mathbf{N}$. The supremum norm on $\widehat{C}^k(X)$ induces the following **seminorm on** $\widehat{H}^k(X)$ by

$$\forall_{\varphi \in \widehat{H}^k(X)} \quad \|\varphi\|_\infty := \inf\{\|f\|_\infty \mid f \in \widehat{C}^k(X), \delta(f) = 0, [f] = \varphi\}. \qquad \diamond$$

Apart from some special cases it is not known whether this seminorm is a norm.

2.2 ■ Elementary properties of bounded cohomology

As we will see later, bounded cohomology is not as tame as singular cohomology, because bounded cohomology fails to satisfy the excision axiom. But at least it is natural, homotopy invariant and fulfils the dimension axiom.

Lemma (2.5) (Naturality). *Let X, Y be topological spaces and $f \colon X \longrightarrow Y$ a continuous map. For each $k \in \mathbf{N}$ the map f induces a homomorphism*

$$\begin{aligned} \widehat{H}^k(f) \colon \widehat{H}^k(Y) &\longrightarrow \widehat{H}^k(X) \\ [\varphi] &\longmapsto \big[c \mapsto \varphi(f \circ \sigma)\big] \end{aligned}$$

of operator norm at most 1. Here composition of f with elements of $C_k(X, \mathbf{R})$ is defined by linear extension of composition with elements of $S_k(X)$.

Proof. Let $g \in \widehat{C}^k(Y)$ with $\delta(g) = 0$. Since g is bounded on $S_k(Y)$, the cochain

$$\begin{aligned} C_k(X, \mathbf{R}) &\longrightarrow \mathbf{R} \\ c &\longmapsto g(f \circ c) \end{aligned}$$

is bounded on $S_k(X)$. Moreover, it follows from the definition of the coboundary operator that this cochain is a cocycle. As the definition of $\widehat{H}^k(f)([g])$ does not depend on the chosen representative g, the map $\widehat{H}^k(f)$ is indeed well-defined. It is clear that $\widehat{H}^k(f)$ is a homomorphism of real vector spaces and that

$$\left\| \widehat{H}^k(f)\big([g]\big) \right\|_\infty \leq \|g\|_\infty .$$

Taking the infimum over all representants yields $\left\| \widehat{H}^k(f)\big([g]\big) \right\|_\infty \leq \|[g]\|_\infty$. □

Lemma (2.6) (Homotopy Invariance). *Bounded cohomology is homotopy invariant, i.e., if $f, g \colon X \longrightarrow Y$ are homotopic maps of topological spaces, then the induced maps are equal:*

$$\widehat{H}^*(f) = \widehat{H}^*(g) \colon \widehat{H}^*(Y) \longrightarrow \widehat{H}^*(X).$$

Proof. Since f and g are homotopic maps, there is a chain homotopy

$$k_* \colon C_*(X, \mathbf{R}) \longrightarrow C_{*+1}(Y, \mathbf{R})$$

between $C_*(f, \mathbf{R})$ and $C_*(g, \mathbf{R})$, i.e.

$$\partial \circ k_{*+1} + k_* \circ \partial = C_{*+1}(f, \mathbf{R}) - C_{*+1}(g, \mathbf{R}).$$

This chain homotopy induces the map

$$\begin{aligned} h^* \colon C^*(Y, \mathbf{R}) &\longrightarrow C^{*-1}(X, \mathbf{R}) \\ c &\longmapsto (-1)^* \cdot c \circ k_{*-1}. \end{aligned}$$

Obviously, h^* maps bounded cochains on bounded cochains, so it restricts to

$$\hat{h}^* := h^*|_{\widehat{C}^*(Y)} \colon \widehat{C}^*(Y) \longrightarrow \widehat{C}^{*-1}(X).$$

Because

$$\begin{aligned} \big(\delta \circ \hat{h}^* + \hat{h}^{*+1} \circ \delta\big)(c) &= \delta((-1)^* \cdot c \circ k_{*-1}) + \hat{h}^{*+1}((-1)^{*+1} \cdot c \circ \partial) \\ &= c \circ k_{*-1} \circ \partial + c \circ \partial \circ k_* \\ &= c \circ \big(C_*(f, \mathbf{R}) - C_*(g, \mathbf{R})\big) \\ &= \big(\widehat{C}^*(f) - \widehat{C}^*(g)\big)(c) \end{aligned}$$

holds for all $c \in \widehat{C}^*(Y)$, it follows that $\hat{h}^*$ is a cochain homotopy between $\widehat{C}^*(f)$ and $\widehat{C}^*(g)$. Hence we get $\widehat{H}^*(f) = \widehat{H}^*(g)$. □

Lemma (2.7) (Dimension Axiom). *Bounded cohomology satisfies the dimension axiom, i.e.*

$$\begin{aligned} & \widehat{H}^0(\bullet) \cong \mathbf{R}, \text{ and} \\ \forall_{k \in \mathbf{N}_{\geq 1}} \quad & \widehat{H}^k(\bullet) = 0, \end{aligned}$$

where $\bullet$ denotes the space consisting of only one point.

Proof. This is the same calculation as in the theory of singular cohomology. □

That the first bounded cohomology group always vanishes will be shown with the duality theorem from 3.1.

2.3 ■ Bounded cohomology of groups

Let G be a discrete group. In [Ivanov, Section 3] the bounded cohomology $\widehat{H}^*(G)$ of G is defined – in analogy to the ordinary cohomology of G via strong relatively injective resolutions of the trivial bounded G-module $\mathbf{R}$.

An interesting fact about bounded cohomology is that it cannot detect amenable groups.

Definition (2.8). Let A be a group and $B(A)$ the set of bounded functions $A \longrightarrow \mathbf{R}$. Then A acts on $B(A)$ from the left via

$$\forall_{f \in B(A)} \quad \forall_{a \in A} \quad a \cdot f := \big(b \mapsto f(ba)\big).$$

The group A is called **amenable**, if there is a linear left-A-invariant function $m \colon B(A) \longrightarrow \mathbf{R}$ such that for all $f \in B(A)$

$$\begin{aligned} \inf_{a \in A} f(a) &\leq m(f), \text{ and} \\ m(f) &\leq \sup_{a \in A} f(a). \end{aligned}$$

◇

Actually, this might seem a mysterious definition. However amenable groups turn out to be quite interesting – for example they are connected with the Banach-Tarski paradox (cf. [Runde, Chapter 0]).

Obviously all finite groups are amenable. Furthermore, all Abelian groups are amenable (compare [Paterson]).

In [Ivanov, Section 3.8] the following theorem is proved:

Theorem (2.9). *Let A be a normal amenable subgroup of the discrete group G. Then the map $\widehat{H}^*(G/A) \longrightarrow \widehat{H}^*(G)$ induced by the canonical homomorphism $G \longrightarrow G/A$ is an (isometric) isomorphism.*

Furthermore, as one might suspect, $\widehat{H}^*(G) \cong \widehat{H}^*(BG)$, if BG is a model of the classifying space of the discrete group G. In fact, much more holds, as is explained in the next section.

2.4 ■ Bounded cohomology and the fundamental group

Perhaps surprisingly, the rather small differences between the definitions of singular and bounded cohomology (of spaces) lead to significant differences in the behaviour of both theories. For instance bounded cohomology only depends on the fundamental group. In particular, if the fundamental group is amenable, the bounded cohomology is equal to the bounded cohomology of the one point space (since bounded cohomology cannot detect amenable groups).

Theorem (2.10). *Let X be a simply connected countable CW-space. Then the bounded cohomology of X is canonically isometrically isomorphic to the bounded cohomology of the one point space.*

Theorem (2.11). *Let X be a countable CW-complex. Then $\widehat{H}^*(X)$ is canonically isomorphic with $\widehat{H}^*\big(\pi_1(X)\big)$.*

Actually these theorems remain true if we drop the condition that the space must be countable cellular. This situation is treated in [Gromov].

Quite understandable proofs of these two theorems can be found in [Ivanov]. We will sketch those proofs: In the following we consider $\widehat{C}^j(X)$ as the set of bounded functions $S_j(X) \longrightarrow \mathbf{R}$.

To prove the first theorem a contracting chain homotopy

$$\mathbf{R} \xleftarrow{K_0} \widehat{C}^0(X) \xleftarrow{K_1} \widehat{C}^1(X) \xleftarrow{K_2} \cdots$$

with $\|K_n\| \leq 1$ for all $n \in \mathbf{N}$ is explicitly constructed by a clever induction argument (involving principal bundles and Eilenberg Mac Lane spaces). Here it is needed that the space is a countable CW-complex to ensure that its homotopy groups are all countable. This implies that there is a model of the Eilenberg Mac Lane space $K\big(\pi_n(X), n\big)$ which is a topological Abelian group.

In order to prove the second theorem the first theorem is applied to show that (where $p\colon \widetilde{X} \longrightarrow X$ is the universal covering of X)

$$0 \longrightarrow \mathbf{R} \longrightarrow \widehat{C}^0(\widetilde{X}) \longrightarrow \widehat{C}^1(\widetilde{X}) \longrightarrow \cdots$$

is a strong relatively injective resolution of the trivial bounded $\pi_1(X)$-module $\mathbf{R}$. Thus the bounded cohomology of the discrete group $\pi_1(X)$ can be computed as the cohomology of the complex

$$0 \longrightarrow \widehat{C}^0\big(\widetilde{X}^{\pi_1(X)}\big) \longrightarrow \widehat{C}^1\big(\widetilde{X}^{\pi_1(X)}\big) \longrightarrow \cdots$$

The induced map $p^*\colon \widehat{C}^*(X) \longrightarrow \widehat{C}^*\big(\widetilde{X}\big)$ establishes an isomorphism between $\widehat{C}^*(X)$ and $\widehat{C}^*(X)^{\pi_1(X)}$ commuting with the differentials. Hence the bounded cohomology of $\pi_1(X)$ coincides with the cohomology of the complex

$$0 \longrightarrow \widehat{C}^0(X) \xrightarrow{\ \partial\ } \widehat{C}^1(X) \xrightarrow{\ \partial\ } \cdots$$

which is the bounded cohomology of the space X.

Theorem (2.12) (Mapping Theorem for Bounded Cohomology). *Let X_1 and X_2 be countable CW-complexes and let $f\colon X_1 \longrightarrow X_2$ a continuous map. If the homomorphism $\pi_1(f)\colon \pi_1(X_1) \longrightarrow \pi_1(X_2)$ is surjective and its kernel is amenable, then the induced homomorphism*

$$\widehat{H}^*(f) = \widehat{\pi_1(f)}\colon \widehat{H}^*(X_2) \longrightarrow \widehat{H}^*(X_1)$$

is an isometric isomorphism.

Proof. A somewhat lengthy computation shows that $\widehat{H}^*(f) = \widehat{\pi_1(f)}$ holds. Hence the theorem is a consequence of Theorem (2.9) and Theorem (2.11). □

2.5 ■ The first bounded cohomology group

As already mentioned, the first bounded cohomology group always vanishes.

Theorem (2.13). *Let G be a discrete group. Then*

$$\widehat{H}^1(G) = 0.$$

Proof. By definition, (compare [Ivanov] for details) a bounded 1-cocycle in the standard complex for the calculation of $\widehat{H}^*(G)$ is a bounded homomorphism $G \longrightarrow \mathbf{R}$. Since there are no bounded nontrivial subgroups of the additive group $\mathbf{R}$, the first bounded cohomology group $\widehat{H}^1(G)$ vanishes. □

Theorem (2.11) implies:

Corollary (2.14). *Let X be a CW-space with countably many cells. Then*

$$\widehat{H}^1(X) = 0.$$

But there is also a geometric proof – using the duality techniques from Section 3.1:

Proof. (Geometric). The Universal Coefficient Theorem yields

$$H_1(X) \otimes \mathbf{R} \cong H_1(X, \mathbf{R}).$$

In other words: if $\alpha \in H_1(X, \mathbf{R})$, then there exist $k \in \mathbf{N}, r_1, \dots, r_k \in \mathbf{R}$ and $\alpha_1, \dots, \alpha_k \in H_1(X)$ such that

$$\alpha = \sum_{j=1}^{k} r_j \cdot \alpha_j.$$

By the Hurewicz Theorem each of the α_j can be represented by a continuous function $\sigma\colon S^1 \longrightarrow X$. For each $j \in \{1, \dots, k\}$ and each $d \in \mathbf{N}_{\geq 1}$ we write $\sigma_j^{(d)}$ for the composition

$$\sigma_j \circ (z \mapsto z^d)\colon S^1 \longrightarrow X.$$

Now the Hurewicz Theorem tells us that

$$[\sigma_j^{(d)}] = d \cdot [\sigma_j]$$

holds in $H_1(X)$. In particular,

$$\|\alpha\|_1 \leq \left\| \sum_{j=1}^{k} r_j \cdot \frac{\sigma^{(d)}}{d} \right\|_1 \leq \frac{1}{d} \cdot \sum_{j=1}^{k} |r_j|.$$

Since this is true for any $d \in \mathbf{N}_{\geq 1}$, we get

$$\|\alpha\|_1 = 0.$$

Now let $\varphi \in \widehat{H}^1(X)$ and $f \in \widehat{C}^1(X)$ a cocycle representing φ. Thus we get from Lemma (3.1) that

$$\forall_{\alpha \in H_1(X,\mathbf{R})} \quad 0 = \varphi(\alpha).$$

By the Universal Coefficient Theorem,

$$\begin{aligned} H^1(X,\mathbf{R}) &\longrightarrow \hom\big(H_1(X),\mathbf{R}\big) \\ [f] &\longmapsto \big([a] \mapsto f(a)\big) \end{aligned}$$

is an isomorphism. Hence the comparison map $i^1 \colon \widehat{H}^1(X) \longrightarrow H^1(X,\mathbf{R})$ is trivial. Thus there is a cochain $u \in C^0(X,\mathbf{R})$ with

$$\delta(u) = f.$$

However, in general u is not bounded. Using u we construct a bounded cochain $\widehat{u} \in \widehat{C}^0(X)$ satisfying $\delta(\widehat{u}) = f$: Let $(X)_{i\in I}$ be the path components of X. For each $i \in I$ we choose a point $x_i \in X_i$. If $x \in X$ then there is exactly one $i(x) \in I$ such that $x \in X_{i(x)}$. We define $\widehat{u} \in \widehat{C}^0(X)$ via

$$\forall_{x\in X} \quad \widehat{u}(x) := u(x) - u(x_{i(x)}).$$

Since x and $x_{i(x)}$ lie in the same path component, there is a path σ_x in X joining these points. So

$$\begin{aligned} \big|\widehat{u}(x)\big| &= \big|u(\sigma_x(0)) - u(\sigma_x(1))\big| \\ &= \big|u(\partial(\sigma_x))\big| \\ &= \big|\delta(u)(\sigma_x)\big| \\ &= \big|f(\sigma_x)\big| \\ &\le \|f\|_\infty \end{aligned}$$

which implies $\widehat{u} \in \widehat{C}^0(X)$. Furthermore we have for all $\sigma \in S_1(X)$

$$\begin{aligned} \delta(\widehat{u})(\sigma) &= \widehat{u}(\sigma(1) - \sigma(0)) \\ &= \widehat{u}(\sigma(1)) - \widehat{u}(\sigma(0)) \\ &= u(\sigma(1)) - u(x_{i(\sigma(1))}) - u(\sigma(0)) - u(x_{i(\sigma(0))}). \end{aligned}$$

Here $i(\sigma(1)) = i(\sigma(0))$, because $\sigma(1)$ and $\sigma(0)$ lie in the same path component. Hence

$$\begin{aligned} \delta(\widehat{u})(\sigma) &= u(\sigma(1)) - u(\sigma(0)) \\ &= u(-\partial(\sigma)) \\ &= \delta(u)(\sigma) \\ &= f(\sigma). \end{aligned}$$

This proves $\varphi = [f] = 0$ in $\widehat{H}^*(X)$. $\square$

2.6 ■ Bounded cohomology and excision

Since $\pi_1(S^1)$ is Abelian, we get from Theorem (2.9) that $\widehat{H}^* \left(S^1\right) \cong \widehat{H}^* (\bullet)$. But it can be shown that $\widehat{H}^2 \left(S^1 \vee S^1\right)$ is not even finitely generated. The "proof" in [Brooks] gives good intuition about this fact, but is not entirely correct – a correct version can be found in [Grigorchuk], [Mitsumatsu].

In particular, bounded cohomology does not satisfy excision and is not always trivial.

Moreover, this result shows that there is no obvious way to describe bounded cohomology using cellular complexes: the space $S^1 \vee S^1$ has no open cells of dimension at least 2, i.e. the cellular complex vanishes in dimension at least 2. However $\widehat{H}^2 \left(S^1 \vee S^1\right)$ is not trivial.

3 ■ Applying Bounded Cohomology

We will now apply the theory of bounded cohomology to compute the simplicial volume. In the first part the duality between $\|\cdot\|_1$ and $\|\cdot\|_\infty$ is explored. In the second part it is shown that the simplicial volume is (up to a constant) compatible with taking products of manifolds.

3.1 ■ Duality

The following lemma explains the duality between $\|\cdot\|_1$ and $\|\cdot\|_\infty$ in our setting. It is the key for applying bounded cohomology to investigate the simplicial volume.

Lemma (3.1). *Let X be a topological space, $k \in \mathbf{N}$ and $\alpha \in H_k(X, \mathbf{R})$.*

1. *Then $\|\alpha\|_1 = 0$ if and only if*

$$\forall_{\varphi \in \widehat{H}^k(X)} \quad \varphi(\alpha) = 0.$$

2. *If $\|\alpha\|_1 > 0$, then*

$$\frac{1}{\|\alpha\|_1} = \inf\big\{\|\varphi\|_\infty \;\big|\; \varphi \in \widehat{H}^k(X), \varphi(\alpha) = 1\big\}.$$

Before we can prove the lemma, we have to introduce the Kronecker product on bounded cohomology:

First we recall the definition of the Kronecker product in the singular theory.

Definition (3.2). Let X be a topological space, R a commutative ring with unit and $k \in \mathbf{N}$. Furthermore, let $f \in C^k(X,R)$ be a cocycle and $a \in C_k(X,R)$ a cycle. Then the **Kronecker product** of $[f] \in H^k(X,R)$ and $[a] \in H_k(X,R)$ is defined by

$$[f]([a]) := f(a). \qquad \diamond$$

In fact, the relations between the coboundary and the boundary operator imply that the Kronecker product is well-defined.

Definition (3.3). Let X be a topological space. The inclusion $\widehat{C}^*(X) \subset C_*(X,\mathbf{R})$ induces a homomorphism $i^* \colon \widehat{H}^*(X) \longrightarrow H^*(X,\mathbf{R})$, which is called **comparison map**. $\diamond$

Definition (3.4). Let X be a topological space, $k \in \mathbf{N}$ and $\varphi \in \widehat{H}^k(X), \alpha \in H_k(X,\mathbf{R})$. Then the **Kronecker product** of φ and α is defined to be

$$\varphi(\alpha) := \big(i^k(\varphi)\big)(\alpha). \qquad \diamond$$

Now we are able to give the proof of the above duality statement:

Proof. (of Lemma (3.1)). Let α be represented by the cycle $a = \sum_{\sigma\in S_k(X)}^{\text{fin}} a_\sigma \cdot \sigma \in C_k(X,\mathbf{R})$. Assume that there is a $\varphi \in \widehat{H}^k(X)$ represented by the bounded cochain $f \in \widehat{C}^k(X)$ with $\varphi(\alpha) = 1$. Then, by definition,

$$\begin{aligned} 1 = |\varphi(\alpha)| &= \left| f\Big(\sum_{\sigma\in S_k(X)}^{\text{fin}} a_\sigma \cdot \sigma\Big)\right| \leq \sum_{\sigma\in S_k(X)}^{\text{fin}} |a_\sigma| \cdot |f(\sigma)| \\ &\leq \|f\|_\infty \cdot \|a\|_1 . \end{aligned}$$

If we take the infimum over all representants of φ and α, it follows that

$$1 \leq \|\varphi\|_\infty \cdot \|\alpha\|_1 .$$

In particular, if $\|\alpha\|_1 = 0$, there can be no such φ.

Now, let $\|\alpha\|_1 > 0$, and let $a \in C_k(X,\mathbf{R})$ be a cycle representing α. We denote the closure with respect to the norm $\|\cdot\|_1$ by $\overline{\cdot}^1$. Thus

$$a \notin \overline{\operatorname{im}\big(\partial \colon C_{k+1}(X,\mathbf{R}) \to C_k(X,\mathbf{R})\big)}^1 .$$

Hence the Hahn-Banach theorem yields the existence of a continuous functional $\widetilde{f}\colon \overline{C_k(X,\mathbf{R})}^1 \longrightarrow \mathbf{R}$ with

$$\begin{aligned} \widetilde{f}(a) &= 1, \\ \widetilde{f}|_{\overline{\operatorname{im}\left(\partial\colon C_{k+1}(X,\mathbf{R})\to C_k(X,\mathbf{R})\right)}^1} &= 0, \\ \|\widetilde{f}\|_\infty &= \frac{1}{\|a\|_1}. \end{aligned}$$

The second property implies that $f := \widetilde{f}|_{C_k(X,\mathbf{R})} \in C^k(X,\mathbf{R})$ is a cocycle. Moreover, $\|f\|_\infty \leq \|\widetilde{f}\|_\infty < \infty$, which implies $f \in \widehat{C}^k(X)$. By construction we have

$$[f](\alpha) = [f]\big([a]\big) = 1$$

and

$$\|[f]\|_\infty \leq \|f\|_\infty \leq \|\widetilde{f}\|_\infty \leq \frac{1}{\|a\|_1} \leq \frac{1}{\|\alpha\|_1}.$$

In particular, if $\|\alpha\|_1 > 0$, then there is a $\varphi \in \widehat{C}^k(X)$ with $\varphi(\alpha) = 1$. Hence, if $\|\alpha\|_1 > 0$,

$$\frac{1}{\|\alpha\|_1} = \inf\big\{\|\varphi\|_\infty \;\big|\; \varphi \in \widehat{H}^k(X), \varphi(\alpha) = 1\big\}. \qquad \square$$

Corollary (3.5). *Let M be an oriented closed connected n-dimensional manifold.*

1. *Then $\|M\| = 0$ if and only if the homomorphism*

   $$i^n\colon \widehat{H}^n(M) \longrightarrow H^n(M,\mathbf{R})$$

   is trivial.

2. *If $\|M\| > 0$, then*

   $$\frac{1}{\|M\|} = \big\|[M]^{\mathbf{R}}\big\|_\infty,$$

   where $[M]^{\mathbf{R}} \in H^n(M,\mathbf{R})$ denotes the image of the cohomological fundamental class of M under the change of rings homomorphism.

Proof. 1. By definition, $[M]^{\mathbf{R}}\big([M]_{\mathbf{R}}\big) = 1$. Furthermore,

$$H^n(M,\mathbf{R}) \cong \mathbf{R} \cdot [M]^{\mathbf{R}}.$$

Hence $\operatorname{im}(i^n) \neq 0$ if and only if $[M]^{\mathbf{R}} \in \operatorname{im}(i^n)$. Therefore the first part of Lemma (3.1) yields that $\|M\| = \|[M]_{\mathbf{R}}\|_1 = 0$ if and only if i^n is trivial.

2. Let $\|M\| > 0$. By the first part, $i^n \colon \widehat{H}^n(M) \longrightarrow H^n(M, \mathbf{R}) \cong [M]^{\mathbf{R}} \cdot \mathbf{R}$ is not trivial and hence surjective. Moreover,

$$\begin{aligned}
\|\varphi\|_\infty &= \inf\big\{ \|f\|_\infty \,\big|\, f \in \widehat{C}^n(X), \delta(f) = 0, [f] = \varphi \text{ in } \widehat{H}^n(M) \big\} \\
&\geq \inf\big\{ \|f\|_\infty \,\big|\, f \in C^n(X, \mathbf{R}), \delta(f) = 0, [f] = i^n(\varphi) \text{ in } H^n(M, \mathbf{R}) \big\} \\
&= \|i^n(\varphi)\|_\infty
\end{aligned}$$

holds for all $\varphi \in \widehat{H}^n(M)$. Using the surjectivity of i^n we obtain (by the second part of Lemma (3.1))

$$\begin{aligned}
\frac{1}{\|M\|} &= \frac{1}{\big\|[M]_{\mathbf{R}}\big\|_1} \\
&= \inf\big\{ \|\varphi\|_\infty \,\big|\, \varphi \in \widehat{H}^n(M), \varphi([M]_{\mathbf{R}}) = 1 \big\} \\
&\geq \inf\big\{ \|i^n(\varphi)\|_\infty \,\big|\, \varphi \in \widehat{H}^n(M), \big(i^n(\varphi)\big)([M]_{\mathbf{R}}) = 1 \big\} \\
&\geq \inf\big\{ \|\psi\|_\infty \,\big|\, \psi \in H^n(M, \mathbf{R}) \cong [M]^{\mathbf{R}} \cdot \mathbf{R}, \psi([M]_{\mathbf{R}}) = 1 \big\} \\
&= \big\|[M]^{\mathbf{R}}\big\|_\infty.
\end{aligned}$$

In particular, $\big\|[M]^{\mathbf{R}}\big\|_\infty$ is finite. On the other hand we get for each $\varepsilon \in \mathbf{R}_{>0}$ a cocycle $f \in C^n(M, \mathbf{R})$ with $[f] = [M]^{\mathbf{R}}$ in $H^n(M, \mathbf{R})$ and

$$\|f\|_\infty \leq \big\|[M]^{\mathbf{R}}\big\|_\infty + \varepsilon < \infty.$$

Thus $f \in \widehat{C}^n(M)$. Let $\varphi \in \widehat{H}^n(M)$ be the corresponding bounded cohomology class. Then

$$\|\varphi\|_\infty \leq \|f\|_\infty < \big\|[M]^{\mathbf{R}}\big\|_\infty + \varepsilon$$

and

$$\varphi([M]_{\mathbf{R}}) = \big(i^n(\varphi)\big)([M]_{\mathbf{R}}) = [f]([M]_{\mathbf{R}}) = [M]^{\mathbf{R}}([M]_{\mathbf{R}}) = 1$$

which implies

$$\frac{1}{\|M\|} = \frac{1}{\big\|[M]_{\mathbf{R}}\big\|_1} \leq \|\varphi\|_\infty \leq \big\|[M]^{\mathbf{R}}\big\|_\infty + \varepsilon. \qquad \square$$

We will now present some immediate consequences of this corollary and the mapping theorem for bounded cohomology (2.12):

Theorem (3.6). *1. Let X be a connected CW-complex with countably many cells and fundamental group G. Then the classifying map $f\colon X \longrightarrow BG$ induces (for all $k \in \mathbf{N}$) an isometric isomorphism*

$$\widehat{H}^k(f)\colon \widehat{H}^k(BG) \longrightarrow \widehat{H}^k(X).$$

2. Let X be a connected CW-complex with countably many cells and amenable fundamental group. Then

$$\widehat{H}^*(X) \cong \widehat{H}^*(\bullet).$$

Proof. 1. From the construction of the classifying map f it follows directly that $\pi_1(f)$ is an isomorphism. Hence the mapping theorem (2.12) proves the first part.

2. This follows from the first part and Theorem (2.9), Theorem (2.10). ☐

Corollary (3.7). *1. Let M be an oriented closed connected manifold of dimension n with fundamental group G and classifying map $f\colon M \longrightarrow BG$. Then*

$$\|M\| = \|H_n(f,\mathbf{R})([M]_{\mathbf{R}})\|_1.$$

2. Moreover, if $G = \pi_1(M)$ is amenable, then

$$\|M\| = 0.$$

Proof. 1. Because $\widehat{H}^n(f)$ is an isometric isomorphism, it follows that

$$\begin{aligned}
&\big\{\|\varphi\|_\infty \;\big|\; \varphi \in \widehat{H}^n(BG), \varphi\big(H_n(f,\mathbf{R})([M]_{\mathbf{R}})\big) = 1\big\} \\
=&\big\{\big\|\widehat{H}^n(f)(\varphi)\big\|_\infty \;\big|\; \varphi \in \widehat{H}^n(BG), \big(\widehat{H}^n(f)(\varphi)\big)([M]_{\mathbf{R}}) = 1\big\} \\
=&\big\{\|\psi\|_\infty \;\big|\; \psi \in \widehat{H}^n(M), \psi([M]_{\mathbf{R}}) = 1\big\}
\end{aligned}$$

Application of Lemma (3.1) proves the first part.

2. The second part of Theorem (3.6) yields that $\widehat{H}^n(M)$ vanishes. Hence the first part of Lemma (3.1) implies that $\|M\| = \big\|[M]_{\mathbf{R}}\big\|_1 = 0.$ ☐

In particular, it follows again that $\big\|S^n\big\| = 0$ for all $n \in \mathbf{N}_{\geq 1}$.

Lemma (3.8). *Let M and N be oriented closed connected manifolds of dimension n and $f\colon M \longrightarrow N$ be a continuous map of degree 1. Then $\pi_1(f)$ is surjective. If the kernel of $\pi_1(f)$ is amenable, then*

$$\|M\| = \|N\|.$$

Proof. Let $p\colon \overline{N} \longrightarrow N$ be the covering of N associated with the subgroup $\operatorname{im}\pi_1(f) \subset \pi_1(N)$. Since $\operatorname{im}\pi_1(f) \subset \operatorname{im}\pi_1(p)$, there is a lift $\overline{f}\colon M \longrightarrow \overline{N}$ of f. From $\deg(f) = 1$ we conclude that

$$H_n(p) \circ H_n(\overline{f}) H_n(f)\colon H_n(M) \longrightarrow H_n(N)$$

is surjective, i.e. $H_n(\overline{N}) \neq 0$. So $\overline{N}$ must be compact. In particular

$$1 = \deg(f) = \deg(p \circ \overline{f}) = \deg(p) \cdot \deg(\overline{f}),$$

which implies $|\deg(p)| = 1$. On the other hand we know (cf. [Dold, Proposition VIII.4.7]), that the number of sheets of p is the degree of p, i.e. p is a one-sheeted covering. Hence

$$\big[\pi_1(N) : \pi_1(\operatorname{im}\pi_1(p))\big] = 1.$$

Therefore $\operatorname{im}\pi_1(f) = \operatorname{im}\pi_1(p) = \pi_1(N)$.

The Mapping Theorem (2.12) implies that $\widehat{H}^*(f)\colon \widehat{H}^*(N) \longrightarrow \widehat{H}^*(M)$ is an isometric isomorphism. Therefore

$$\begin{aligned}
&\{\|\varphi\|_\infty \mid \varphi \in \widehat{H}^n(N)\varphi([N]_{\mathbf{R}}) = 1\}\\
=&\{\|\widehat{H}^n(f)(\varphi)\|_\infty \mid \varphi \in \widehat{H}^n(N), \varphi\big(H_n(f,\mathbf{R})([M]_{\mathbf{R}})\big) = 1\}\\
=&\{\|\widehat{H}^n(f)(\varphi)\|_\infty \mid \varphi \in \widehat{H}^n(N), \big(\widehat{H}^n(f)(\varphi)\big)([M]_{\mathbf{R}}) = 1\}\\
=&\{\|\psi\|_\infty \mid \psi \in \widehat{H}^n(M), \psi([M]_{\mathbf{R}}) = 1\}.
\end{aligned}$$

Thus Lemma (3.1) proves the equality $\|M\| = \|N\|$. □

3.2 ■ Products of Manifolds

Using bounded cohomology we are able to prove that the simplicial volume is compatible with products (up to a constant):

Theorem (3.9). *Let M and N be oriented closed connected manifolds.*

1. *Then (with the constant that will be introduced in Lemma (3.13))*

$$\|N \times M\| \leq c\big(\dim(N) + \dim(M)\big) \cdot \|N\| \cdot \|M\|.$$

2. *On the other hand,*

$$\|N\| \cdot \|M\| \leq \|N \times M\|.$$

The proof uses explicit descriptions of the cup-product and the cross-products in singular (co-)homology. So we first recall some basic properties of these products:

- *Homological Cross-Product.* Let X and Y be topological spaces, $p, q \in \mathbf{N}$ and $\sigma \in S_p(X), \tau \in S_q(Y)$. Then the homological cup-product $\sigma \times \tau \in S_{p+q}(X \times Y)$ is defined as follows – using the shuffle homomorphism.

  Definition (3.10). A (p, q)**-shuffle** is a pair (μ, ν) of disjoint sets of integers

  $$1 \leq \mu_1 < \mu_2 < \cdots < \mu_p \leq p + q \text{ and } 1 \leq \nu_1 < \nu_2 < \cdots < \nu_q \leq p + q$$

  between 1 and $p+q$. The set of all (p, q)-shuffles is denoted by $S_{p,q}$. Let $\operatorname{sgn}(\mu, \nu)$ be the sign of the permutation $(\mu_1, \ldots, \mu_p, \nu_1, \nu_q)$.

  We define the linear map $\eta^\mu \colon \Delta^{p+q} \longrightarrow \Delta^p$ by

  $$\forall_{j \in \{0, \ldots, p\}} \quad \forall_{k \in \{\mu_j, \ldots, \mu_{j+1}\}} \quad \eta^\mu(e^k) := e^j,$$

  where $\mu_0 := 0, \mu_{p+1} := p + q + 1$. Analogously, $\eta^\nu \colon \Delta^{p+q} \longrightarrow \Delta^q$ is defined. $\diamondsuit$

  Definition (3.11). The **cross-product of** $\sigma \in S_p(X)$ **and** $\tau \in S_q(Y)$ is given by

  $$\sigma \times \tau := \sum_{(\mu,\nu) \in S_{p,q}} \operatorname{sgn}(\mu, \nu) \cdot (\sigma \circ \eta^\mu, \tau \circ \eta^\nu).$$

  This yields – by linear extension and passage to homology – the **homological cross-product**

  $$\times \colon H_p(X, \mathbf{R}) \times H_q(Y, \mathbf{R}) \longrightarrow H_{p+q}(X \times Y, \mathbf{R}). \qquad \diamondsuit$$

  In fact, the homological cross-product is well-defined (cf. [Dold]).

  Definition (3.12). We define for all $k \in \mathbf{N}$

  $$c(k) := \max\{|S_{p,q}| \mid p, q \in \mathbf{N} \text{ with } p + q = k\}. \qquad \diamondsuit$$

  In particular,

  $$\|\sigma \times \tau\|_1 \leq c(p+q)$$

  for all $\sigma \in S_p(X)$ and all $\tau \in S_q(Y)$.

Lemma (3.13). *For all topological spaces X, Y all $p, q \in \mathbf{N}$ and all homology classes $\alpha \in H_p(X, \mathbf{R}), \beta \in H_q(Y, \mathbf{R})$ we have*

$$\|\alpha \times \beta\|_1 \leq c(p+q) \cdot \|\alpha\|_1 \cdot \|\beta\|_1 .$$

Proof. Let α and β be represented by the two cycles $\sum^{\text{fin}}_{\sigma \in S_p(X)} a_\sigma \cdot \sigma$ and $\sum^{\text{fin}}_{\tau \in S_q(Y)} b_\tau \cdot \tau$, respectively. Then, by definition,

$$\begin{aligned}
\|\alpha \times \beta\|_1 &= \left\| \left[\sum^{\text{fin}}_{\sigma \in S_p(X)} a_\sigma \cdot \sigma \right] \times \left[\sum^{\text{fin}}_{\tau \in S_q(Y)} b_\tau \cdot \tau \right] \right\|_1 \\
&= \left\| \sum^{\text{fin}}_{\sigma \in S_p(X)} \sum^{\text{fin}}_{\tau \in S_q(Y)} a_\sigma b_\tau \cdot [\sigma \times \tau] \right\|_1 \\
&\leq \sum^{\text{fin}}_{\sigma \in S_p(X)} \sum^{\text{fin}}_{\tau \in S_q(Y)} |a_\sigma| \cdot |b_\tau| \cdot \left\| [\sigma_j \times \tau_k] \right\|_1 \\
&\leq c(p+q) \cdot \sum^{\text{fin}}_{\sigma \in S_p(X)} |a_\sigma| \cdot \sum^{\text{fin}}_{\tau \in S_q(Y)} |b_\tau| \\
&= c(p+q) \cdot \left\| \sum^{\text{fin}}_{\sigma \in S_p(X)} a_\sigma \cdot \sigma \right\|_1 \cdot \left\| \sum^{\text{fin}}_{\tau \in S_q(Y)} b_\tau \cdot \tau \right\|_1 .
\end{aligned}$$

Taking the infimum over all representants of α and β proves the lemma. □

- *Cohomological Cross-Product.* We will only need the following property of the cohomological cross-product:

  Lemma (3.14). *Let X, Y be topological spaces, $p, q \in \mathbf{N}$ and $\varphi \in H^p(X, \mathbf{R}), \psi \in H^q(Y, \mathbf{R})$. Then the cohomological cross-product satisfies the following equality:*

  $$\forall_{\alpha \in H_p(X, \mathbf{R})} \quad \forall_{\beta \in H_q(Y, \mathbf{R})} \quad \big(\varphi \times \psi\big)(\alpha \times \beta) = (-1)^{\deg(\psi) \deg(\alpha)} \cdot \varphi(\alpha) \cdot \psi(\beta).$$

  Proof. A proof can be found in [Bredon, Theorem VI 5.4]. □

- *Cup-Product.*

  Definition (3.15). Let X be a topological space, $p, q \in \mathbf{N}$ and $\sigma \in S_{p+q}(X)$. Then the p**-front face of** σ, denoted by $\sigma\rfloor_p \in S_p(X)$ is defined as the affine extension of

  $$\forall_{j \in \{0, \dots, p\}} \quad \sigma\rfloor_p := \sigma(e_j).$$

Analogously, the q**-back face of** σ, written as ${}_q\lfloor\sigma \in S_q(X)$ is given by

$$\forall_{j\in\{0,\dots,q\}} \quad {}_q\lfloor\sigma(e_j) := \sigma(e_{n-q+j}). \qquad \diamond$$

Definition (3.16). Let X be a topological space, $p, q \in \mathbf{N}$. Let $f \in C^p(X, \mathbf{R})$ and $g \in C^q(X, \mathbf{R})$. Then the **cup-product**, $f \cup g \in C^{p+q}(X, \mathbf{R})$, is given by

$$\forall_{\sigma\in S_{p+q}(X)} \quad (f \cup g)(\sigma) := (-1)^{pq} \cdot f(\sigma\rfloor_p) \cdot g({}_q\lfloor\sigma). \qquad \diamond$$

Passing to cohomology this induces the **cup-product**

$$\cup\colon H^p(X, \mathbf{R}) \times H^q(X, \mathbf{R}) \longrightarrow H^{p+q}(X, \mathbf{R}).$$

Then we have the following relation between the cup-product and the cohomological cross-product:

Lemma (3.17). *Let X, Y be topological spaces and $p_X\colon X \times Y \longrightarrow X$, $p_Y\colon X \times Y \longrightarrow Y$ the natural projections. Then*

$$H^*(p_X, \mathbf{R})(\varphi) \cup H^*(p_Y, \mathbf{R})(\psi) = \varphi \times \psi$$

for all $\varphi \in C^*(X, \mathbf{R})$ and all $\psi \in C^*(Y, \mathbf{R})$.

Proof. A proof is, for example, given in [Bredon, p. 327]. ☐

- *Cup-Product for Bounded Cohomology.* It is easy to construct a cup-product in bounded cohomology:

Lemma (3.18). *Let X be a topological space and $p, q \in \mathbf{N}$.*

1. *If $f \in \widehat{C}^p(X)$ and $g \in \widehat{C}^q(X)$, then*

$$f \cup g \in \widehat{C}^{p+q}(X).$$

2. *The map $\cup\colon \widehat{C}^p(X) \times \widehat{C}^q(X) \longrightarrow \widehat{C}^{p+q}(X)$ induces a map*

$$\cup\colon \widehat{H}^p(X) \times \widehat{H}^q(X) \longrightarrow \widehat{H}^{p+q}(X)$$

in bounded cohomology. This map is also called ***cup-product****.*

3. *Let $\beta \in \widehat{H}^p(X)$ and $\gamma \in \widehat{H}^q(X)$. Then*

$$\|\beta \cup \gamma\|_\infty \le \|\beta\|_\infty \cdot \|\gamma\|_\infty.$$

Proof. 1. The construction of the ordinary cup-product yields

$$\|f \cup g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty ,$$

i.e. the cochain $f \cup g$ is bounded and lies hence in $\widehat{C}^{p+q}(X)$.

2. The same calculation as in singular cohomology shows that the map $\cup \colon \widehat{H}^p(X) \times \widehat{H}^q(X) \longrightarrow \widehat{H}^{p+q}(X)$ is well-defined.

3. Let $f \in \widehat{C}^p(X)$ and $g \in \widehat{C}^q(X)$ be representants of β and γ, respectively. Then (as in the first part)

$$\|\beta \cup \gamma\|_\infty \leq \|f \cup g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty .$$

Taking the infimum over all representants of β and γ yields the desired inequality. □

Proof. (of Theorem (3.9)). Let $n := \dim N, m := \dim M$ and $k := n + m$.

1. Since the homology of N vanishes in dimensions higher than n and since the homology of M vanishes in dimensions higher than m and $H_n(N, \mathbf{Z}) \cong \mathbf{Z} \cong H_m(M, \mathbf{Z})$, the cross-product

$$\times \colon H_n(N, \mathbf{Z}) \otimes H_m(M, \mathbf{Z}) \longrightarrow H_{n+m}(N \times M, \mathbf{Z})$$

is – by the Künneth-Theorem – an isomorphism. Hence $[N] \times [M]$ is a generator of $H_{n+m}(N \times M, \mathbf{Z})$.

This implies

$$\begin{aligned} \|N \times M\| &= \left\| \left([N] \times [M]\right)_{\mathbf{R}} \right\|_1 \\ &= \|[N]_{\mathbf{R}} \times [M]_{\mathbf{R}}\|_1 , \end{aligned}$$

and application of lemma (3.13) results in the claimed inequality

$$\begin{aligned} \|N \times M\| &\leq c(n+m) \cdot \|[N]_{\mathbf{R}}\|_1 \cdot \|[M]_{\mathbf{R}}\|_1 \\ &= c(n+m) \cdot \|N\| \cdot \|M\| . \end{aligned}$$

2. If $\|N\| = 0$ or $\|M\| = 0$, then the inequality obviously holds. We therefore assume that $\|N\| > 0$ and $\|M\| > 0$. Hence, by lemma (3.1) for each $\varepsilon \in \mathbf{R}_{>0}$, there are $\beta \in \widehat{H}^n(N)$ and $\gamma \in \widehat{H}^m(M)$ with

$$\begin{aligned} \beta([N]_{\mathbf{R}}) = 1,\ \|\beta\|_\infty &\leq \frac{1}{\|N\|} + \varepsilon \text{ and} \\ \gamma([M]_{\mathbf{R}}) = 1,\ \|\gamma\|_\infty &\leq \frac{1}{\|M\|} + \varepsilon. \end{aligned}$$

Let

$$\begin{aligned}\beta_N &:= \widehat{H}^*\,(p_N)(\beta) \in \widehat{H}^{n+m}\,(N \times M)\\ \gamma_M &:= \widehat{H}^*\,(p_M)(\gamma) \in \widehat{H}^{n+m}\,(N \times M),\end{aligned}$$

where $p_N \colon N \times M \longrightarrow N$ and $p_M \colon N \times M \longrightarrow M$ are the canonical projections. Then by Lemma (3.17) and Lemma (3.14) we have

$$\begin{aligned}\big(\beta_N \cup \gamma_M\big)([N]_{\mathbf{R}} \times [M]_{\mathbf{R}}) &= \big(\beta \times \gamma\big)([N]_{\mathbf{R}} \times [M]_{\mathbf{R}})\\ &= (-1)^{nm} \cdot \beta([N]_{\mathbf{R}}) \cdot \gamma([M]_{\mathbf{R}})\\ &= (-1)^{nm}.\end{aligned}$$

Furthermore, by Lemma (3.18).3,

$$\begin{aligned}\|\beta_N \cup \gamma_M\|_\infty &\leq \|\beta_N\|_\infty \cdot \|\gamma_M\|_\infty\\ &\leq \|\beta\|_\infty \cdot \|\gamma\|_\infty\\ &\leq \left(\frac{1}{\|N\|} + \varepsilon\right) \cdot \left(\frac{1}{\|M\|} + \varepsilon\right)\end{aligned}$$

holds. Thus Lemma (3.1) implies $\|N\| \cdot \|M\| \leq \|N \times M\|$. □

4 ■ Other Volumes

In this section we will give some examples of manifolds with nonzero simplicial volume. Moreover the relation with other volumes is explored. To do so some notions from differential geometry are required.

A friendly introduction to the Ricci calculus is given in [Jänich]. The book [Lee] contains very good geometric descriptions of geodesics and (sectional) curvature.

4.1 ■ Straightening

The simplicial volume is a topological invariant. So it is not obvious how differential geometry comes into play. But in the special case of manifolds with nonpositive sectional curvature one can use "good" simplices to calculate the simplicial volume:

Let M be an orientable closed connected Riemannian manifold with nonpositive sectional curvature and $p\colon \widetilde{M} \longrightarrow M$ its universal Riemannian covering (i.e. p is an orientation preserving local isometry). For $k \in \mathbf{N}$ and $y_0, \dots, y_k \in \widetilde{M}$ the **geodesic k-simplex** $\sigma_{y_0,\dots,y_k}\colon \Delta^k \longrightarrow \widetilde{M}$ is inductively defined as follows:

Since $\widetilde{M}$ has also nonpositive sectional curvature, the Cartan-Hadamard Theorem (cf. [Lee, Theorem 11.5]) implies that for each two points $p_0, p_1 \in \widetilde{M}$ there is exactly one geodesic $\gamma\colon [0,1] \longrightarrow \widetilde{M}$ with

$$\gamma(0) = p_0 \text{ and } \gamma(1) = p_1.$$

If $t \in [0,1]$, we use the notation

$$t \cdot p_0 + (1-t) \cdot p_1 := \gamma(t).$$

Since γ is unique, this is well-defined.

If $k = 0$ we define

$$\begin{aligned} \sigma_{y_0} \colon \Delta^0 &\longrightarrow \widetilde{M} \\ 1 &\longmapsto y_0. \end{aligned}$$

If we assume by induction that $\sigma_{y_0,\dots,y_{k-1}} \colon \Delta^{k-1} \longrightarrow \widetilde{M}$ is already constructed, we let

$$\begin{aligned} \sigma_{y_0,\dots,y_k} \colon \Delta^k &\longrightarrow \widetilde{M} \\ (1-t) \cdot s + t \cdot (0,\dots,0,1) &\longmapsto (1-t) \cdot \sigma_{y_0,\dots,y_{k-1}}(s) + t \cdot y_k \end{aligned}$$

(where $\Delta^{k-1} \subset \Delta^k$ is identified with the set $\{(x_0,\dots,x_k) \in \Delta^k \mid x_k = 0\}$).

It can be shown that $\sigma_{y_0,\dots,y_k}$ is continuous and thus defines a singular simplex in $\widetilde{M}$.

Definition (4.1). Let M be as above and $k \in \mathbf{N}$. A singular simplex $\sigma \in S_k(M)$ is called **straight** if there exist $y_0, \dots, y_k \in \widetilde{M}$ such that

$$\sigma = p \circ \sigma_{y_0,\dots,y_k}.$$

A singular chain is called **straight** if it is the linear combination of straight simplices.

We will denote the set of straight k-simplices of M by $\mathsf{S}_k(M)$ and the set of straight k-chains of M by $\mathsf{C}_k(M)$. ◇

The following lemma shows that we can restrict ourselves to straight chains:

Lemma (4.2). *Let M be as above. Then there is a chain map*

$$S \colon C_*(M, \mathbf{R}) \longrightarrow \mathsf{C}_*(M) \subset C_*(M, \mathbf{R})$$

which is chain homotopic to $\mathrm{id}_{C_*(M,\mathbf{R})}$ *and which satisfies*

$$\forall_{c \in C_*(M,\mathbf{R})} \quad \|S(c)\|_1 \leq \|c\|_1 .$$

Proof. Let $k \in \mathbf{N}$ and $\sigma \in S_k(M)$. Since Δ^k is simply connected, there is a lift $\widetilde{\sigma} \colon \Delta^k \longrightarrow \widetilde{M}$. Let $\overline{\sigma}$ be the geodesic k-simplex having the same vertices as $\widetilde{\sigma}$, i.e.

$$\overline{\sigma} = \sigma_{\widetilde{\sigma}(e_0),\dots,\widetilde{\sigma}(e_k)}.$$

Now, we define

$$S(\sigma) := p \circ \overline{\sigma}.$$

It is not difficult to see that this definition does not depend on the chosen lift $\widetilde{\sigma}$.

Obviously S linearly extends to a map $C_*(M, \mathbf{R}) \longrightarrow C_*(M)$ and (by construction)

$$\forall_{c \in C_*(M,\mathbf{R})} \quad \|S(c)\|_1 \leq \|c\|_1 .$$

The map S is a chain map, since: Let $\sigma \in S_k(M)$. Then the inductive construction of straight simplices shows that $\overline{\sigma \circ \partial_j} = \overline{\sigma} \circ \partial_j$ for all $j \in \{0, \dots, k\}$ which implies

$$\begin{aligned} S(\partial(\sigma)) &= \sum_{j=0}^{k} (-1)^j S(\sigma \circ \partial_j) = \sum_{j=0}^{k} (-1)^j \overline{\sigma \circ \partial_j} \\ &= \sum_{j=0}^{k} (-1)^j \overline{\sigma} \circ \partial_j = \sum_{j=0}^{k} (-1)^j S(\sigma) \circ \partial_j \\ &= \partial\big(S(\sigma)\big). \end{aligned}$$

Why are S and $\mathrm{id}_{C_*(M,\mathbf{R})}$ chain homotopic? If $\sigma \in S_k(M)$, then

$$\begin{aligned} \Delta^k \times [0,1] &\longrightarrow M \\ (t,s) &\longmapsto p\big(s \cdot \widetilde{\sigma}(t) + (1-s) \cdot \overline{\sigma}(t)\big) \end{aligned}$$

is a homotopy between σ and $S(\sigma)$ relative to the vertices. Using the same method as in the proof of homotopy invariance of singular homology one can construct a chain homotopy between S and $\mathrm{id}_{C_*(M,\mathbf{R})}$. □

Corollary (4.3). *Let M be as above and $k \in \mathbf{N}$. Then for all $\alpha \in H_k(M, \mathbf{R})$*

$$\|\alpha\|_1 = \inf\big\{ \|c\|_1 \;\big|\; c \in \operatorname{im} S \text{ represents } \alpha \big\}.$$

I.e. we only need to consider straight simplices for the computation of the simplicial volume. □

4.2 ■ Algebraic volume

If an orientable connected closed manifold is triangulated, this triangulation yields a representant of the homological fundamental class (as explained in [Bredon, p. 338f.]). If we sum up the volumes of the images of the singular simplices of this representant we get the volume of the manifold.

But this obviously does not hold for each representant of the homological fundamental class, since we have to consider multiplicities. A workaround is the algebraic volume for manifolds with nonpositive sectional curvature.

Let M be an orientable closed connected Riemannian manifold with nonpositive sectional curvature.

Definition (4.4). Let M be as above and $n := \dim M$. For $\sigma \in \mathsf{S}_n(M)$ the **algebraic volume** is defined as follows:

If there is one point $t \in \Delta^n$ such that $d_t\overline{\sigma}$ is orientation preserving, then this is true for all points in Δ^n, and vice versa. Hence

$$\operatorname{algvol}(\sigma) := \begin{cases} \operatorname{vol}\big(\overline{\sigma}(\Delta^n)\big) & \text{if } d\overline{\sigma} \text{ is orientation-preserving} \\ -\operatorname{vol}\big(\overline{\sigma}(\Delta^n)\big) & \text{if } d\overline{\sigma} \text{ is orientation-reversing} \end{cases}$$

is well-defined. By linear extension we get

$$\operatorname{algvol}\colon \mathsf{S}_k(M) \longrightarrow \mathbf{R}. \qquad \diamond$$

Definition (4.5). A Riemannian manifold is called **hyperbolic** if it is complete and its sectional curvature is everywhere -1. $\diamond$

Theorem (4.6). *Let M be an oriented closed connected hyperbolic manifold and $a \in \mathsf{C}_*(M)$ a straight cycle representing the fundamental class $[M]_{\mathbf{R}}$. Then*

$$\operatorname{algvol}(a) = \operatorname{vol}(M).$$

Proof. A proof can be found in the proof of [Benedetti-Petronio, Proposition C.4.6]. $\square$

It is possible to generalise the proof given in [Benedetti-Petronio] to orientable closed connected Riemannian manifolds with nonpositive sectional curvature.

4.3 ■ Hyperbolic manifolds

If $n \in \mathbf{N}$ the volume $\operatorname{vol}(\sigma)$ of all geodesic simplices σ in hyperbolic n-space $\mathbf{H}^n$ is bounded uniformly by a real constant.

Definition (4.7). For $n \in \mathbf{N}$ we write

$$v_n := \sup\big\{\operatorname{vol}(\sigma) \,\big|\, \sigma \in \mathsf{S}_n(\mathbf{H}^n)\big\}. \qquad \diamond$$

Theorem (4.8). *Let M be an oriented closed connected hyperbolic manifold of dimension n. Then*

$$\|M\| \geq \frac{\operatorname{vol}(M)}{v_n}.$$

Proof. By Lemma (4.2) there is a straight cycle $a = \sum^{\text{fin}}_{\sigma \in \mathsf{S}_n(M)} a_\sigma \cdot \sigma \in \mathsf{C}_n(M)$ representing $[M]_{\mathbf{R}}$. Then Theorem (4.6) yields (since M is hyperbolic, the hyperbolic n-space $\mathbf{H}^n$ is the universal covering space of M)

$$\begin{aligned}
\operatorname{vol}(M) &= |\operatorname{algvol}(a)| \\
&= \left|\operatorname{algvol}\left(\sum^{\text{fin}}_{\sigma \in \mathsf{S}_n(M)} a_\sigma \cdot \sigma\right)\right| \\
&\leq \sum^{\text{fin}}_{\sigma \in \mathsf{S}_n(M)} |a_\sigma| \cdot |\operatorname{algvol}(\sigma)| \\
&= \sum^{\text{fin}}_{\sigma \in \mathsf{S}_n(M)} |a_\sigma| \cdot |\operatorname{vol}(\overline{\sigma}(\Delta^n))| \\
&\leq v_n \cdot \|a\|_1 .
\end{aligned}$$

If we now take the infimum over all straight cycles representing $[M]_{\mathbf{R}}$ then we get from Corollary (4.3) that $\|M\| \geq \operatorname{vol}(M)/v_n$. □

Corollary (4.9). *In particular there are manifolds whose simplicial volume does not vanish.* □

The inequality in the above theorem is actually an equality (Theorem C.4.2 in [Benedetti-Petronio]). Another generalisation of the above theorem is the following:

Theorem (4.10). *Let $n \in \mathbf{N}$. Then there is a constant $C_n \in \mathbf{R}_{>0}$ such that the following holds: Let $\delta \in \mathbf{R}_{>0}$ and M an n-dimensional orientable closed connected Riemannian manifold whose sectional curvature is bounded from above by $-\delta$. Then*

$$\|M\| \geq C_n \cdot \delta^{n/2} \cdot \operatorname{vol}(M).$$

Proof. A proof is given in [Inoue, Yano]. □

4.4 ■ Compact surfaces

We will calculate the simplicial volume of oriented closed connected surfaces of genus at least 2, i.e. we will show that in dimension 2 the inequality of Theorem (4.8) actually is an equality. A good description of the hyperbolic structure on hyperbolic surfaces is given in [Benedetti-Petronio, Section B.3].

Theorem (4.11). *Let M be an orientable closed connected smooth surface of genus $g \geq 2$. Then*

$$\|M\| = 4g - 4.$$

If M is hyperbolic, then

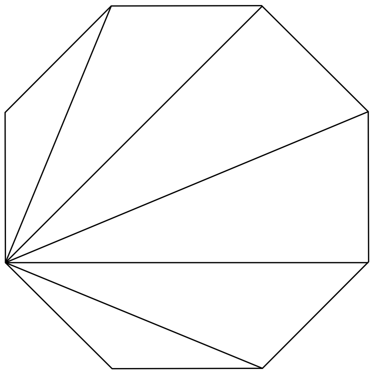
$$\|M\| = \frac{\mathrm{vol}(M)}{v_2}.$$

To prove this theorem we need the following inequality:

Lemma (4.12). *Let $g \in \mathbf{N}_{\geq 2}$ and F_g be an oriented closed connected surface of genus g. Then*

$$\|F_g\| \leq 4g - 2.$$

Proof. The surface F_g can be constructed from a regular $4g$-gon by identifying certain edges (compare [Massey]). This $4g$-gon can be obviously triangulated by $4g-2$ simplices:



This triangulation induces a "triangulation" of F_g consisting of $4g-2$ simplices $\sigma_1, \dots, \sigma_{4g-2}$. This is no triangulation of F_g in the strict sense, but its second barycentric subdivision is. Hence we get from [Bredon, p. 338f.] that $\sum_{j=1}^{4g-2} \sigma_j$ is a cycle representing $[F_g]_{\mathbf{R}}$, which implies the inequality

$$\|F_g\| \leq 4g - 2. \qquad \square$$

Proof. (of Theorem (4.11)) For each $d \in \mathbf{N}_{\geq 2}$ there is d-sheeted covering

$$p_d \colon M_d \longrightarrow M$$

with connected covering space: By [Massey, Example IV 5.3] we have

$$\begin{aligned} \pi_1(F_g) &\cong \left\langle a_1, \dots, a_g, b_1, \dots, b_g \,\middle|\, \prod_{j=1}^{g} [a_j, b_j] = 1 \right\rangle, \\ \pi_1(F_g)^{\mathrm{ab}} &\cong \mathbf{Z}^{2g}. \end{aligned}$$

Thus there is a surjective group homomorphism

$$\pi_1(M) \cong \pi_1(F_g) \longrightarrow \pi_1(F_g)^{\mathrm{ab}} \cong \mathbf{Z}^{2g} \longrightarrow \mathbf{Z}.$$

The preimage under this homomorphism of a subgroup of $\mathbf{Z}$ of index d is a subgroup of $\pi_1(M)$ of index d. Therefore M admits a d-sheeted covering $p_d \colon M_d \longrightarrow M$ (the covering corresponding to this subgroup).

Hence M_d also is an oriented closed connected surface. As one can see using a CW-decomposition (or by calculating the volume – if the manifolds are hyperbolic),

$$\chi(M_d) = d \cdot \chi(M).$$

Thus we get for the genus

$$\begin{aligned} g(M_d) &= \frac{-\chi(M_d) + 2}{2} = \frac{-d \cdot \chi(M) + 2}{2} \\ &= d \cdot g(M) - d + 1, \end{aligned}$$

i.e. M_d and $F_{d \cdot g(M) - d + 1}$ are homotopy equivalent (compare [Massey]). Applying Lemma (1.16) and Lemma (4.12) results in

$$\begin{aligned} \|M\| &= \frac{1}{d} \cdot \|M_d\| = \frac{1}{d} \cdot \left\| F_{g(M_d)} \right\| \\ &\leq \frac{1}{d} \cdot \bigl(4g(M_d) - 2\bigr) \\ &= 4g - 4 + \frac{2}{d}. \end{aligned}$$

Since this inequality holds for all $d \in \mathbf{N}_{\geq 2}$, it follows that

$$\|M\| \leq 4g - 4.$$

In view of the homotopy invariance of the simplicial volume we may assume that M is hyperbolic (since there is a hyperbolic model of the surface with genus $g(M)$). Hence the Gauß-Bonnet Theorem yields

$$\operatorname{vol}(M) = 2\pi \cdot \chi(M) = 2\pi \cdot (2g(M) - 2).$$

Moreover, $v_2 = \pi$ (compare the proof of [Benedetti-Petronio, Lemma C.2.3]). Hence Theorem (4.8) implies

$$\|M\| = 4g - 4. \qquad \square$$

4.5 ■ The proportionality principle of simplicial volume

One of the most interesting facts on the simplicial volume is the following proportionality principle.

Theorem (4.13) (Proportionality Principle). *Let M and N be oriented closed connected Riemannian manifolds such that there universal Riemannian coverings are isometrically isomorphic. Then*

$$\frac{\|M\|}{\operatorname{vol} M} = \frac{\|N\|}{\operatorname{vol} N}.$$

Unfortunately, there are only two proofs of this wonderful theorem in the literature: one by Thurston (cf. [Thurston, Chapter 6]) using "smearing", and one by Gromov (in [Gromov]) which is rather unreadable.

In both proofs a "new" simplicial volume is defined. However, Thurston does not prove that his new volume is the same as the usual simplicial volume. Gromov shows that his new simplicial volume is the same as the old one, but the rest of the proof is not completely worked out.

5 ■ Simplicial Volume and L^2-Invariants

In this section some evidence for the truth of Gromov's conjecture is collected. The main reason to believe that the conjecture is true is the fact that the simplicial volume and L^2-Betti numbers show similar behaviour.

Since aspherical spaces are the central objects of concern, we recall the definition:

Definition (5.1). A space is called **aspherical** if all its higher homotopy groups are zero. ◇

5.1 ■ The zeroth L^2-Betti number

A rather easy argument shows, that the conjecture is true for the zeroth L^2-Betti number:

Proposition (5.2). *Let M be an aspherical connected closed manifold. Then the fundamental group $\pi_1(M)$ is torsion free and hence infinite.*

Proof. Assume that $\pi_1(M)$ contains a finite cyclic subgroup G. Since M is aspherical and of finite dimension n, the cellular chain complex $C_*^{\text{cell}}(\widetilde{M}; \mathbf{Z})$ is a bounded projective resolution of the trivial $\mathbf{Z}G$-module $\mathbf{Z}$. This implies that the group homology

$$H_*(G; \mathbf{Z}) \cong H_*\big(C_*^{\text{cell}}(M; \mathbf{Z}) \otimes_{\mathbf{Z}G} \mathbf{Z}\big)$$

vanishes in all degrees greater then n.

On the other hand it is known that the group homology of finite cyclic groups is periodic (cf. [Weibel, Theorem 6.2.3]). Hence $\pi_1(M)$ cannot contain a finite cyclic subgroup, i.e. $\pi_1(M)$ is torsion free. □

Corollary (5.3). *If M is an aspherical closed connected manifold, then*

$$b_0^{(2)}(\widetilde{M}) = \frac{1}{|\pi_1(M)|} = 0.$$

Proof. This follows directly from [Lück, Theorem 6.54(8)]. □

5.2 ■ Gromov's conjecture

Looking at the conjecture and the known evidence one could also suspect, that the converse is true: If M is an oriented aspherical connected closed manifold whose L^2-Betti numbers vanish, then $\|M\| = 0$.

But this statement is false: If M is an oriented closed connected hyperbolic manifold of odd dimension, then

$$b_p^{(2)}(\widetilde{M}) = 0$$

for all $p \in \mathbf{N}$ (cf. [Lück, Theorem 1.54]). On the other hand $\|M\| \neq 0$ as we have seen in Section 4.3.

5.3 ■ Proportionality Principles

One of the most striking similarities is the proportionality principle. It is valid for L^2-Betti numbers (and L^2-torsion) and for the simplicial volume (as already mentioned in Section 4.5):

Theorem (5.4) (Proportionality for L^2-Invariants). *Let M be a simply connected Riemannian manifold. Then there are for each $p \in \mathbf{N}$ constants $B_p^{(2)} \in \mathbf{R}_{>0}$ and $T^{(2)} \in \mathbf{R}_{>0}$ with the following property: For any discrete group G acting freely and cocompactly on M by isometries the following holds for all $p \in \mathbf{N}$*

$$b_p^{(2)}(M; \mathcal{N}(G)) = B_p^{(2)} \cdot \operatorname{vol}(G \setminus M)$$

If M with this G-action is of determinant class, then

$$\rho^{(2)}(M, \mathcal{N}(G)) = T^{(2)} \cdot \operatorname{vol}(G \setminus M).$$

A proof can be found in [Lück, Theorem 3.183].

5.4 ■ Multiplicativity

As we have seen in Section 1.3 the simplicial volume is multiplicative with respect to the mapping degree. So Gromov's conjecture predicts, that if M is an aspherical oriented closed connected manifold admitting a self-map of degree at least two, then all L^2-Betti numbers of M vanish. This statement is known to be true if this self-map is in fact a covering (see [Lück, Example 1.37]).

Theorem (5.5). *Let M be an aspherical oriented closed connected smooth manifold such that there exists a continuous map $f\colon M \longrightarrow M$ with the following properties:*

- *The degree satisfies $|\deg(f)| \geq 2$.*
- *There is an $N \in \mathbf{N}$ such that $\pi_1(f^n)|_{\operatorname{im}\pi_1(f^n)}$ is injective for all $n \in \mathbf{N}_{\geq N}$.*

Then

$$\forall_{p\in\mathbf{N}} \quad b_p^{(2)}\big(\widetilde{M}\big) = 0.$$

To prove the theorem we need an estimate of L^2-Betti numbers.

Lemma (5.6). *Let $g\colon N_1 \longrightarrow N_2$ be a continuous map of oriented closed connected smooth n-dimensional manifolds with $\deg(g) \neq 0$. Assume that N_1 admits a finite CW-structure with $e_p(N_1)$ cells of dimension p. Then*

$$\forall_{p\in\mathbf{N}} \quad b_p^{(2)}\big(\widetilde{N}_2\big) \leq e_p(N_1).$$

Proof. The proof of this lemma is contained in the proof of [Lück, Theorem 14.40]. □

Proof. For each $n \in \mathbf{N}$ there is a covering $p_n\colon M_n \longrightarrow M$ of M associated to the subgroup $\operatorname{im}\pi_1(f^n)$. Since $\operatorname{im}\pi_1(p) = \operatorname{im}\pi_1(f^n)$ there is in particular a lift $\overline{f^n}\colon M \longrightarrow M_n$ with $p_n \circ \overline{f^n} = f^n$. Thus

$$\begin{aligned}
\big(\deg(f)\big)^n &= \deg(f^n) \\
&= \deg(p_n)\cdot\deg\big(\overline{f^n}\big) \\
&= \big[\pi_1(M) : \operatorname{im}\pi_1(p_n)\big]\cdot\deg\big(\overline{f^n}\big) \\
&= \big[\pi_1(M) : \operatorname{im}\pi_1(f^n)\big]\cdot\deg\big(\overline{f^n}\big).
\end{aligned}$$

Now $\deg(f) \neq 0$ implies

$$\big[\pi_1(M) : \operatorname{im}\pi_1(f^n)\big] \leq \big|\deg(f)\big|^n < \infty.$$

By [Lück, Theorem 1.35(9)] we conclude

$$\forall_{p\in\mathbf{N}} \quad b_p^{(2)}\big(\widetilde{M}_n\big) = \big[\pi_1(M) : \operatorname{im}\pi_1(f^n)\big] \cdot b_p^{(2)}\big(\widetilde{M}\big).$$

Since M is smooth and compact, M admits a CW-structure with finitely many cells. For $p \in \mathbf{N}$ we denote the number of p-cells of this CW-structure by $e_p(M)$.

Applying the above lemma yields

$$b_p^{(2)}\big(\widetilde{M}\big) \leq \frac{e_p(M)}{[\pi_1(M) : \operatorname{im}\pi_1(f^n)]}$$

for all $p, n \in \mathbf{N}$.

Assume that there is an $m \in \mathbf{N}$ such that

$$\forall_{n\in\mathbf{N}_{\geq m}} \quad \operatorname{im}\pi_1(f^n) = \operatorname{im}\pi_1(f^m).$$

By the premise of the theorem this implies that the restriction

$$\pi_1(f^n)|_{\operatorname{im}\pi_1(f^m)} \colon \operatorname{im}\pi_1(f^m) \longrightarrow \operatorname{im}\pi_1(f^m)$$

is bijective for all $n \in \mathbf{N}$ with $n \geq \max(N, m)$.

For $n \in \mathbf{N}$ we write $F_n := \overline{f^n} \circ p_n$, i.e. we get the following commutative diagram:

$$\begin{tikzcd} M_n \arrow[r, "F_n"] \arrow[d, "p_n"'] & M_n \arrow[d, "p_n"] \\ M \arrow[r, "f_n"'] \arrow[ur, "\overline{f^n}"] & M \end{tikzcd}$$

If $n \in \mathbf{N}_{\geq\max(N,m)}$, then $\pi_1(F_n)\colon \pi_1(M_n) \longrightarrow \pi_1(M_n)$ is an isomorphism (because $\pi_1(p_n)\colon \pi_1(M_n) \longrightarrow \operatorname{im}\pi_1(f^n)$ is an isomorphism). Since M is an aspherical finite CW-complex the covering space M_n is also an aspherical finite CW-complex (the number of sheets is finite). Therefore – by the Whitehead Theorem – it follows that F_n is a homotopy equivalence. Since M_n is moreover an oriented closed connected manifold (M is an oriented closed connected manifold), this implies $|\deg(F_n)| = 1$. Hence

$$\begin{aligned} 1 = |\deg(F_n)| &= \big|\deg\big(\overline{f^n}\big)\big| \cdot |\deg(p_n)| = \big|\deg(f^n)\big| = \big|\deg(f)\big|^n \\ &\geq 2^n, \end{aligned}$$

which is a contradiction. Thus there cannot be such an m. In particular

$$\big[\operatorname{im}\pi_1(f^n) : \operatorname{im}\pi_1(f^{n+1})\big] \geq 2$$

for all $n \in \mathbf{N}$, which proves

$$\begin{aligned}
\forall_{p\in\mathbf{N}} \quad b_p^{(2)}(\widetilde{M}) &\leq \inf_{n\in\mathbf{N}} \frac{e_p(M)}{\big[\pi_1(M) : \operatorname{im}\pi_1(f^n)\big]} \\
&= \frac{e_p(M)}{\big[\pi_1(M) : \operatorname{im}\pi_1(f)\big]} \cdot \inf_{n\in\mathbf{N}} \frac{e_p(M)}{\big[\operatorname{im}\pi_1(f) : \operatorname{im}\pi_1(f^n)\big]} \\
&\leq \frac{e_p(M)}{\big[\pi_1(M) : \operatorname{im}\pi_1(f)\big]} \cdot \inf_{n\in\mathbf{N}} \frac{1}{2^n} \\
&= 0.
\end{aligned}$$

□

Definition (5.7). A group G is called **Hopfian** if every epimorphism $G \longrightarrow G$ is an isomorphism. ◇

Examples for Hopfian groups are finite groups , the integers $\mathbf{Z}$ and more general residually finite groups. The group $\ell^\infty(\mathbf{N}, \mathbf{R})$ is not Hopfian as can be seen by considering the shift operator.

The proof of the above theorem yields:

Corollary (5.8). *Let M be an aspherical oriented closed connected smooth manifold. If there is a continuous map $f\colon M \longrightarrow M$ such that $|\deg(f)| \geq 2$ and if all subgroups of $\pi_1(M)$ of finite index are Hopfian, then*

$$\forall_{p\in\mathbf{N}} \quad b_p^{(2)}(M) = 0.$$

5.5 ■ S^1-operations

Furthermore, the simplicial volume and the L^2-Betti numbers behave similarly under S^1-operations:

Theorem (5.9). *Let M be an oriented closed connected smooth manifold with a non-trivial S^1-action. Then $\|M\| = 0$.*

Proofs are given in [Gromov, p. 41] and [Yano].

Theorem (5.10). *Let M be an aspherical closed smooth manifold with non-trivial S^1-action. Then*

$$\forall_{p\in\mathbf{N}} \quad b_p^{(2)}(\widetilde{M}) = 0.$$

This is proven in [Lück, Corollary 1.43].

References

[Benedetti-Petronio] R. Benedetti, C. Petronio. *Lectures on Hyperbolic Geometry*, Springer 1992.

[Bredon] G.E. Bredon. *Topology and Geometry*, Springer 1995 (Corrected second printing).

[Brooks] R. Brooks. Some remarks on bounded cohomology. In *Ann. Math. Studies*, 97, p. 53−63, 1981.

[Dold] A. Dold, *Lectures on Algebraic Topology*, Springer 1972.

[Grigorchuk] R.I. Grigorchuk. Some results on bounded cohomology. In *Combinatorial and geometric group theory (Edinburgh, 1993)*, p. 111−163. Cambridge University Press, Cambridge, 1995.

[Gromov] M. Gromov. Volume and bounded cohomology. In *Publ. Math. IHES*, 56, p. 5−99, 1982.

[Gromov91] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex 1991)*, pages 1 − 295. Cambridge University Press, Cambridge, 1993.

[Inoue, Yano] H. Inoue, K. Yano. The Gromov invariant of negatively curved manifolds. In *Topology*, 21, No. 1, p. 83−89, 1981.

[Ivanov] N.V. Ivanov. Foundations of the theory of bounded cohomology. In *J. Soviet Math.*, 37, p. 1090−1114, 1987.

[Jänich] K. Jänich. *Vector Analysis*, Springer 2001.

[Lee] J.M. Lee. *Riemannian Manifolds: An Introduction to Curvature*, Springer, 1997.

[Lück] W. Lück. *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, Volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*, Springer, 2002.

[Massey] W.S. Massey. *Algebraic Topology: An Introduction*, Springer, 1967.

[Mitsumatsu] Y. Mitsumatsu. Bounded cohomology and ℓ^1-homology of surfaces. In *Topology*, 23, No. 4, p. 465−471, 1984.

[Paterson] A.L.T. Paterson. *Amenability*, AMS, Providence, RI, 1988.

[Runde] V. Runde. *Lectures on amenability*, Volume 1774 of *Lecture Notes in Mathematics*, Springer, 2002.

[Thurston] W.P. Thurston. *Geometry and Topology of 3-Manifolds*, lecture notes, Princeton, 1978. Available at `http://www.msri.org/publications/books/gt3m`.

[Weibel] C.A. Weibel. *An Introduction to Homological Algebra*, Cambridge University Press, Cambridge, 1994.

[Yano] K. Yano. Gromov invariant and S^1-actions. In *J. Fac. Sci. U. Tokyo, Sec. 1A Math.*, 29, No. 3, p. 493−501, 1982.